

# forall $x$

Calgary Remix

*Solutions to Selected Exercises*

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## CHAPTER 1

# *Arguments*

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So **I should take my sunglasses.**
2. **It must have been sunny.** I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. **You're the culprit!**
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence **Colonel Mustard did it in the kitchen with the lead-piping.** Recall, after all, that the gun had not been fired.

## CHAPTER 2

# *Valid arguments*

A. Which of the following arguments is valid? Which is invalid?

1. Socrates is a man.
2. All men are carrots.
- ∴ Socrates is a carrot. Valid

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.
- ∴ Abe Lincoln was born in Illinois. Valid

1. If I pull the trigger, Abe Lincoln will die.
  2. I do not pull the trigger.
  - ∴ Abe Lincoln will not die. Invalid
- Abe Lincoln might die for some other reason: someone else might pull the trigger; he might die of old age.*

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.
- ∴ Abe Lincoln was from France. Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.
- ∴ The world will not end today. Valid

1. Joe is now 19 years old.
  2. Joe is now 87 years old.
  - ∴ Bob is now 20 years old. Valid
- An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. It is impossible for all the premises to be true; so it is certainly impossible that the premises are all true and the conclusion is false.*

B. Could there be:

1. A valid argument that has one false premise and one true premise? **Yes.**  
Example: the first argument, above.
2. A valid argument that has only false premises? **Yes.**  
Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.
3. A valid argument with only false premises and a false conclusion? **Yes.**  
The same example will suffice.
4. An invalid argument that can be made valid by the addition of a new premise? **Yes.**  
Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
5. A valid argument that can be made invalid by the addition of a new premise? **No.**  
An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

## CHAPTER 3

# *Other logical notions*

**A.** For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

**B.** For each of the following: Is it a necessary truth, a necessary falsehood, or contingent?

1. Elephants dissolve in water.
2. Wood is a light, durable substance useful for building things.
3. If wood were a good building material, it would be useful for building things.
4. I live in a three story building that is two stories tall.
5. If gerbils were mammals they would nurse their young.

**C.** Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.  
If you put an elephant in water, it will disintegrate.
2. All mammals dissolve in water.  
If you put an elephant in water, it will disintegrate.
3. George Bush was the 43rd president.  
Barack Obama is the 44th president.

4. Barack Obama is the 44th president.  
Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.  
All mammals dissolve in water.

**D.** Which of the following pairs of sentences are necessarily equivalent?

1. Thelonious Monk played piano.  
John Coltrane played tenor sax.
2. Thelonious Monk played gigs with John Coltrane.  
John Coltrane played gigs with Thelonious Monk.
3. All professional piano players have big hands.  
Piano player Bud Powell had big hands.
4. Bud Powell suffered from severe mental illness.  
All piano players suffer from severe mental illness.
5. John Coltrane was deeply religious.  
John Coltrane viewed music as an expression of spirituality.

**E.** Consider the following sentences:

- G<sub>1</sub> There are at least four giraffes at the wild animal park.  
 G<sub>2</sub> There are exactly seven gorillas at the wild animal park.  
 G<sub>3</sub> There are not more than two Martians at the wild animal park.  
 G<sub>4</sub> Every giraffe at the wild animal park is a Martian.

Now consider each of the following collections of sentences. Which are jointly possible? Which are jointly impossible?

- |   |                    |
|---|--------------------|
| 1. Sentences G <sub>2</sub> , G <sub>3</sub> , and G <sub>4</sub> | Jointly possible   |
| 2. Sentences G <sub>1</sub> , G <sub>3</sub> , and G <sub>4</sub> | Jointly impossible |
| 3. Sentences G <sub>1</sub> , G <sub>2</sub> , and G <sub>4</sub> | Jointly possible   |
| 4. Sentences G <sub>1</sub> , G <sub>2</sub> , and G <sub>3</sub> | Jointly possible   |

**F.** Consider the following sentences.

- M<sub>1</sub> All people are mortal.  
 M<sub>2</sub> Socrates is a person.  
 M<sub>3</sub> Socrates will never die.  
 M<sub>4</sub> Socrates is mortal.

Which combinations of sentences are jointly possible? Mark each “possible” or “impossible.”

1. Sentences M<sub>1</sub>, M<sub>2</sub>, and M<sub>3</sub>
2. Sentences M<sub>2</sub>, M<sub>3</sub>, and M<sub>4</sub>



3. Sentences  $M_2$  and  $M_3$
4. Sentences  $M_1$  and  $M_4$
5. Sentences  $M_1$ ,  $M_2$ ,  $M_3$ , and  $M_4$

**G.** Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument that has one false premise and one true premise  
Yes: 'All whales are mammals (*true*). All mammals are plants (*false*). So all whales are plants.'
2. A valid argument that has a false conclusion  
Yes. (See example from previous exercise.)
3. A valid argument, the conclusion of which is a necessary falsehood  
Yes: ' $1 + 1 = 3$ . So  $1 + 2 = 4$ .'
4. An invalid argument, the conclusion of which is a necessary truth  
No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
5. A necessary truth that is contingent  
No. If a sentence is a necessary truth, it cannot possibly be false, but a contingent sentence can be false.
6. Two necessarily equivalent sentences, both of which are necessary truths  
Yes: ' $4$  is even', ' $4$  is divisible by  $2$ '.
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent  
No. A necessary truth cannot possibly be false, while a contingent sentence can be false. So in any situation in which the contingent sentence is false, it will have a different truth value from the necessary truth. Thus they will not necessarily have the same truth value, and so will not be equivalent.
8. Two necessarily equivalent sentences that together are jointly impossible  
Yes: ' $1 + 1 = 4$ ' and ' $1 + 1 = 3$ '.
9. A jointly possible collection of sentences that contains a necessary falsehood  
No. If a sentence is necessarily false, there is no way to make it true, let alone it along with all the other sentences.
10. A jointly impossible set of sentences that contains a necessary truth  
Yes: ' $1 + 1 = 4$ ' and ' $1 + 1 = 2$ '.

**H.** Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
2. A valid argument with true premises and a false conclusion
3. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
4. A jointly possible collection of sentences, all of which are contingent
5. A false necessary truth

6. A valid argument with false premises
7. A necessarily equivalent pair of sentences that are not jointly possible
8. A necessary truth that is also a necessary falsehood
9. A jointly possible collection of sentences that are all necessary falsehoods

## CHAPTER 5

# Connectives

A. Using the symbolization key given, symbolize each English sentence in TFL.

*M*: Those creatures are men in suits.

*C*: Those creatures are chimpanzees.

*G*: Those creatures are gorillas.

1. Those creatures are not men in suits.

$\neg M$

2. Those creatures are men in suits, or they are not.

$(M \vee \neg M)$

3. Those creatures are either gorillas or chimpanzees.

$(G \vee C)$

4. Those creatures are neither gorillas nor chimpanzees.

$\neg(C \vee G)$

5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.

$(C \rightarrow \neg(G \vee M))$

6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.

$(M \vee (C \vee G))$

B. Using the symbolization key given, symbolize each English sentence in TFL.

*A*: Mister Ace was murdered.

*B*: The butler did it.

*C*: The cook did it.

*D*: The Duchess is lying.

*E*: Mister Edge was murdered.

*F*: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.

$(A \vee E)$

2. If Mister Ace was murdered, then the cook did it.

$(A \rightarrow C)$

3. If Mister Edge was murdered, then the cook did not do it.  
( $E \rightarrow \neg C$ )
4. Either the butler did it, or the Duchess is lying.  
( $B \vee D$ )
5. The cook did it only if the Duchess is lying.  
( $C \rightarrow D$ )
6. If the murder weapon was a frying pan, then the culprit must have been the cook.  
( $F \rightarrow C$ )
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.  
( $\neg F \rightarrow (C \vee B)$ )
8. Mister Ace was murdered if and only if Mister Edge was not murdered.  
( $A \leftrightarrow \neg E$ )
9. The Duchess is lying, unless it was Mister Edge who was murdered.  
( $D \vee E$ )
10. If Mister Ace was murdered, he was done in with a frying pan.  
( $A \rightarrow F$ )
11. Since the cook did it, the butler did not.  
( $C \wedge \neg B$ )
12. Of course the Duchess is lying!  
 $D$

C. Using the symbolization key given, symbolize each English sentence in TFL.

- $E_1$ : Ava is an electrician.  
 $E_2$ : Harrison is an electrician.  
 $F_1$ : Ava is a firefighter.  
 $F_2$ : Harrison is a firefighter.  
 $S_1$ : Ava is satisfied with her career.  
 $S_2$ : Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.  
( $E_1 \wedge E_2$ )
2. If Ava is a firefighter, then she is satisfied with her career.  
( $F_1 \rightarrow S_1$ )
3. Ava is a firefighter, unless she is an electrician.  
( $F_1 \vee E_1$ )
4. Harrison is an unsatisfied electrician.  
( $E_2 \wedge \neg S_2$ )
5. Neither Ava nor Harrison is an electrician.  
 $\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.  
( $(E_1 \wedge E_2) \wedge \neg(S_1 \vee S_2)$ )
7. Harrison is satisfied only if he is a firefighter.

$$(S_2 \rightarrow F_2)$$

8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.

$$((\neg E_1 \rightarrow \neg E_2) \wedge (E_1 \rightarrow E_2))$$

9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.

$$(S_1 \leftrightarrow \neg S_2)$$

10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.

$$((E_2 \wedge F_2) \rightarrow S_2)$$

11. It cannot be that Harrison is both an electrician and a firefighter.

$$\neg(E_2 \wedge F_2)$$

12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.

$$((F_2 \wedge F_1) \leftrightarrow \neg(E_2 \vee E_1))$$

D. Using the symbolization key given, translate each English-language sentence into TFL.

$J_1$ : John Coltrane played tenor sax.

$J_2$ : John Coltrane played soprano sax.

$J_3$ : John Coltrane played tuba

$M_1$ : Miles Davis played trumpet

$M_2$ : Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.

$$J_1 \wedge J_2$$

2. Neither Miles Davis nor John Coltrane played tuba.

$$\neg(M_2 \vee J_3) \text{ or } \neg M_2 \wedge \neg J_3$$

3. John Coltrane did not play both tenor sax and tuba.

$$\neg(J_1 \wedge J_3) \text{ or } \neg J_1 \vee \neg J_3$$

4. John Coltrane did not play tenor sax unless he also played soprano sax.

$$\neg J_1 \vee J_2$$

5. John Coltrane did not play tuba, but Miles Davis did.

$$\neg J_3 \wedge M_2$$

6. Miles Davis played trumpet only if he also played tuba.

$$M_1 \rightarrow M_2$$

7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.

$$M_1 \rightarrow (J_1 \vee (J_2 \vee J_3))$$

8. If John Coltrane played tuba then Miles Davis played neither trumpet nor tuba.

$$J_3 \rightarrow \neg(M_1 \vee M_2) \text{ or } J_3 \rightarrow (\neg M_1 \wedge \neg M_2)$$

9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.

$$(J_3 \wedge M_2) \leftrightarrow (\neg J_1 \wedge \neg M_1) \text{ or } (J_3 \wedge M_2) \leftrightarrow \neg(J_1 \vee M_1)$$

**E.** Give a symbolization key and symbolize the following English sentences in TFL.

- A*: Alice is a spy.  
*B*: Bob is a spy.  
*C*: The code has been broken.  
*G*: The German embassy will be in an uproar.

- Alice and Bob are both spies.  
 $(A \wedge B)$
- If either Alice or Bob is a spy, then the code has been broken.  
 $((A \vee B) \rightarrow C)$
- If neither Alice nor Bob is a spy, then the code remains unbroken.  
 $(\neg(A \vee B) \rightarrow \neg C)$
- The German embassy will be in an uproar, unless someone has broken the code.  
 $(G \vee C)$
- Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.  
 $((C \vee \neg C) \wedge G)$
- Either Alice or Bob is a spy, but not both.  
 $((A \vee B) \wedge \neg(A \wedge B))$

**F.** Give a symbolization key and symbolize the following English sentences in TFL.

- F*: There is food to be found in the pridelands.  
*R*: Rafiki will talk about squashed bananas.  
*A*: Simba is alive.  
*K*: Scar will remain as king.

- If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.  
 $(F \rightarrow R)$
- Rafiki will talk about squashed bananas unless Simba is alive.  
 $(R \vee A)$
- Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.  
 $((R \vee \neg R) \wedge F)$
- Scar will remain as king if and only if there is food to be found in the pridelands.  
 $(K \leftrightarrow F)$
- If Simba is alive, then Scar will not remain as king.  
 $(A \rightarrow \neg K)$

**G.** For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

*P*: Dorothy plays the Piano in the morning.

*C*: Roger wakes up cranky.

*D*: Dorothy is distracted.

$(P \rightarrow C), (P \vee D), (\neg C \rightarrow D)$

2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

*T*<sub>1</sub>: It rains on Tuesday

*T*<sub>2</sub>: It snows on Tuesday

*S*: Neville is sad on Tuesday

*C*: Neville is cold on Tuesday

$(T_1 \vee T_2), (T_1 \rightarrow S), (T_2 \rightarrow C), (S \vee C)$

3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

*Z*: Zoog remembered to do his chores

*C*: Things are clean

*N*: Things are neat

$(Z \rightarrow (C \wedge \neg N)), (\neg Z \rightarrow (N \wedge \neg C)), ((N \vee C) \wedge \neg(N \wedge C)).$

**H.** For each argument, write a symbolization key and translate the argument as well as possible into TFL. The part of the passage in italics is there to provide context for the argument, and doesn't need to be symbolized.

1. It is going to rain soon. I know because my leg is hurting, and my leg hurts if it's going to rain.
2. *Spider-man tries to figure out the bad guy's plan.* If Doctor Octopus gets the uranium, he will blackmail the city. I am certain of this because if Doctor Octopus gets the uranium, he can make a dirty bomb, and if he can make a dirty bomb, he will blackmail the city.
3. *A westerner tries to predict the policies of the Chinese government.* If the Chinese government cannot solve the water shortages in Beijing, they will have to move the capital. They don't want to move the capital. Therefore they must solve the water shortage. But the only way to solve the water shortage is to divert almost all the water from the Yangzi river northward. Therefore the Chinese government will go with the project to divert water from the south to the north.

**I.** We symbolized an *exclusive or* using '∨', '∧', and '¬'. How could you symbolize an *exclusive or* using only two connectives? Is there any way to symbolize an *exclusive or* using only one connective?

For two connectives, we could offer any of the following:

$$\begin{aligned} & \neg(\mathcal{A} \leftrightarrow \mathcal{B}) \\ & (\neg\mathcal{A} \leftrightarrow \mathcal{B}) \\ & (\neg(\neg\mathcal{A} \wedge \neg\mathcal{B}) \wedge \neg(\mathcal{A} \wedge \mathcal{B})) \end{aligned}$$

But if we wanted to symbolize it using only one connective, we would have to introduce a new primitive connective.



## CHAPTER 6

# *Sentences of TFL*

**A.** For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

- |   |                 |
|---|-----------------|
| 1. $(A)$  | (a) no (b) no   |
| 2. $J_{374} \vee \neg J_{374}$                                    | (a) no (b) yes  |
| 3. $\neg\neg\neg\neg F$   | (a) yes (b) yes |
| 4. $\neg \wedge S$  | (a) no (b) no   |
| 5. $(G \wedge \neg G)$  | (a) yes (b) yes |
| 6. $(A \rightarrow (A \wedge \neg F)) \vee (D \leftrightarrow E)$ | (a) no (b) yes  |
| 7. $[(Z \leftrightarrow S) \rightarrow W] \wedge [J \vee X]$      | (a) no (b) yes  |
| 8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \wedge D)$   | (a) no (b) no   |

**B.** Are there any sentences of TFL that contain no atomic sentences? Explain your answer.

**No.** Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, ..., that were ultimately built out of atomic sentences.

**C.** What is the scope of each connective in the sentence

$$[(H \rightarrow I) \vee (I \rightarrow H)] \wedge (J \vee K)$$

The scope of the left-most instance of ' $\rightarrow$ ' is ' $(H \rightarrow I)$ '.

The scope of the right-most instance of ' $\rightarrow$ ' is ' $(I \rightarrow H)$ '.

The scope of the left-most instance of ' $\vee$ ' is ' $[(H \rightarrow I) \vee (I \rightarrow H)]$ '.

The scope of the right-most instance of ' $\vee$ ' is ' $(J \vee K)$ '.

The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

CHAPTER 10

# Complete truth tables

A. Complete truth tables for each of the following:

1.  $A \rightarrow A$

$A$	$A \rightarrow A$
T	T T T
F	F T F

2.  $C \rightarrow \neg C$

$C$	$C \rightarrow \neg C$
T	T F F T
F	F T T F

3.  $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

$A$	$B$	$(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
T	T	T T T T T T F F T
T	F	T F F T F T T T F
F	T	F F T T F F T F T
F	F	F T F T T F F T F

4.  $(A \rightarrow B) \vee (B \rightarrow A)$

$A$	$B$	$(A \rightarrow B) \vee (B \rightarrow A)$
T	T	T T T T T T T
T	F	T F F T F T T
F	T	F T T T T F F
F	F	F T F T F T F

5.  $(A \wedge B) \rightarrow (B \vee A)$

$A$	$B$	$(A \wedge B) \rightarrow (B \vee A)$
T	T	T
T	F	T
F	T	T
F	F	T

6.  $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

$A$	$B$	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
T	T	F
T	F	F
F	T	F
F	F	T

7.  $[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$

$A$	$B$	$C$	$[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

8.  $[(A \wedge B) \wedge C] \rightarrow B$

$A$	$B$	$C$	$[(A \wedge B) \wedge C] \rightarrow B$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

9.  $\neg[(C \vee A) \vee B]$

$A$	$B$	$C$	$\neg[(C \vee A) \vee B]$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

**B.** Check all the claims made in introducing the new notational conventions in §10.3, i.e. show that:

1. ‘ $((A \wedge B) \wedge C)$ ’ and ‘ $(A \wedge (B \wedge C))$ ’ have the same truth table

$A$	$B$	$C$	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

2. ‘ $((A \vee B) \vee C)$ ’ and ‘ $(A \vee (B \vee C))$ ’ have the same truth table

$A$	$B$	$C$	$(A \vee B) \vee C$	$A \vee (B \vee C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

3. ‘ $((A \vee B) \wedge C)$ ’ and ‘ $(A \vee (B \wedge C))$ ’ do not have the same truth table

$A$	$B$	$C$	$(A \vee B) \wedge C$	$A \vee (B \wedge C)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

4. ‘ $((A \rightarrow B) \rightarrow C)$ ’ and ‘ $(A \rightarrow (B \rightarrow C))$ ’ do not have the same truth table

$A$	$B$	$C$	$(A \rightarrow B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

Also, check whether:

5. ‘ $((A \leftrightarrow B) \leftrightarrow C)$ ’ and ‘ $(A \leftrightarrow (B \leftrightarrow C))$ ’ have the same truth table  
Indeed they do:

$A$	$B$	$C$	$(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

C. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1.  $\neg(S \leftrightarrow (P \rightarrow S))$

$\neg$	$(S \leftrightarrow (P \rightarrow S))$
F	T
F	T
F	F
T	F

2.  $\neg[(X \wedge Y) \vee (X \vee Y)]$

$\neg$	$[(X \wedge Y) \vee (X \vee Y)]$
F	T
F	T
F	F
T	F

3.  $(A \rightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

(A	$\rightarrow$	B)	$\leftrightarrow$	( $\neg$	B	$\leftrightarrow$	$\neg$	A)
T	T	T	T	F	T	T	F	T
T	F	F	T	T	F	F	F	T
F	T	T	F	F	T	F	T	F
F	T	F	T	T	F	T	T	F

4.  $[C \leftrightarrow (D \vee E)] \wedge \neg C$

[C	$\leftrightarrow$	(D	$\vee$	E)]	$\wedge$	$\neg$	C
T	T	T	T	T	F	F	T
T	T	T	T	F	F	F	T
T	T	F	T	T	F	F	T
T	F	F	F	F	F	F	T
F	F	T	T	T	F	T	F
F	F	T	T	F	F	T	F
F	F	F	T	T	F	T	F
F	T	F	F	F	T	T	F

5.  $\neg(G \wedge (B \wedge H)) \leftrightarrow (G \vee (B \vee H))$

$\neg$	(G	$\wedge$	(B	$\wedge$	H))	$\leftrightarrow$	(G	$\vee$	(B	$\vee$	H))
F	T	T	T	T	T	F	T	T	T	T	T
T	T	F	T	F	F	T	T	T	T	T	F
T	T	F	F	F	T	T	T	T	F	T	T
T	T	F	F	F	F	T	T	T	F	F	F
T	F	F	T	T	T	T	F	T	T	T	T
T	F	F	T	F	F	T	F	T	T	T	F
T	F	F	F	F	T	T	F	T	F	T	T
T	F	F	F	F	F	F	F	F	F	F	F

**D.** Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1.  $(D \wedge \neg D) \rightarrow G$

(D	$\wedge$	$\neg$	D)	$\rightarrow$	G
T	F	F	T	T	T
T	F	F	T	T	F
F	F	T	F	T	T
F	F	T	F	T	F

2.  $(\neg P \vee \neg M) \leftrightarrow M$

$(\neg$	P	$\vee$	$\neg$	M)	$\leftrightarrow$	M
F	T	F	F	T	T	T
F	T	T	T	F	F	F
T	F	T	F	T	T	T
T	F	T	T	F	T	F

3.  $\neg\neg(\neg A \wedge \neg B)$

$\neg$	$\neg$	$(\neg$	A	$\wedge$	$\neg$	B)
F	T	F	T	F	F	T
F	T	F	T	F	T	F
F	T	T	F	F	F	T
T	F	T	F	T	T	F

4.  $[(D \wedge R) \rightarrow I] \rightarrow \neg(D \vee R)$

$[(D$	$\wedge$	R)	$\rightarrow$	I]	$\rightarrow$	$\neg$	(D	$\vee$	R)
T	T	T	T	T	F	F	T	T	T
T	T	T	F	F	T	F	T	T	T
T	F	F	T	T	F	F	T	T	F
T	F	F	T	F	F	F	T	T	F
F	F	T	T	T	F	F	F	T	T
F	F	T	T	F	F	F	F	T	T
F	F	F	T	T	T	T	F	F	F
F	F	F	T	F	T	T	F	F	F

5.  $\neg[(D \leftrightarrow O) \leftrightarrow A] \rightarrow (\neg D \wedge O)$

$\neg$	$[(D$	$\leftrightarrow$	O)	$\leftrightarrow$	A]	$\rightarrow$	$(\neg$	D	$\wedge$	O)
F	T	T	T	T	T	T	F	T	F	T
T	T	T	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	F	T	F	F
F	T	F	F	T	F	T	F	T	F	F
T	F	F	T	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T	F	T	T
F	F	T	F	T	T	T	T	F	F	F
T	F	T	F	F	F	T	T	F	F	F

If you want additional practice, you can construct truth tables for any of the sentences and arguments in the exercises for the previous chapter.

CHAPTER 11

# *Semantic concepts*

A. Revisit your answers to §10A. Determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

- |   |               |
|---|---------------|
| 1. $A \rightarrow A$  | Tautology     |
| 2. $C \rightarrow \neg C$   | Neither       |
| 3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ | Tautology     |
| 4. $(A \rightarrow B) \vee (B \rightarrow A)$                             | Tautology     |
| 5. $(A \wedge B) \rightarrow (B \vee A)$                                  | Tautology     |
| 6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$                | Tautology     |
| 7. $[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$                      | Contradiction |
| 8. $[(A \wedge B) \wedge C] \rightarrow B$                                | Tautology     |
| 9. $\neg[(C \vee A) \vee B]$  | Neither       |

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1.  $A \rightarrow A, \neg A \rightarrow \neg A, A \wedge A, A \vee A$  Jointly consistent (see line 1)

$A$	$A \rightarrow A$	$\neg A \rightarrow \neg A$	$A \wedge A$	$A \vee A$
T	T T T	F T T F T	T T T	T T T
F	F T F	T F T T F	F F F	F F F

2.  $A \vee B, A \rightarrow C, B \rightarrow C$  Jointly consistent (see line 1)



<i>A</i>	<i>B</i>	<i>C</i>	$A \vee B$	$A \rightarrow C$	$B \rightarrow C$
T	T	T	T <b>T</b> T	T <b>T</b> T	T <b>T</b> T
T	T	F	T <b>T</b> T	T <b>F</b> F	T <b>F</b> F
T	F	T	T <b>T</b> T	T <b>T</b> T	F <b>T</b> T
T	F	F	T <b>F</b> F	T <b>F</b> F	F <b>T</b> F
F	T	T	F <b>T</b> F	F <b>T</b> T	T <b>T</b> T
F	T	F	F <b>T</b> T	F <b>T</b> F	T <b>F</b> F
F	F	T	F <b>F</b> F	F <b>T</b> T	F <b>T</b> T
F	F	F	F <b>F</b> F	F <b>T</b> F	F <b>T</b> F

3.  $B \wedge (C \vee A), A \rightarrow B, \neg(B \vee C)$  Jointly inconsistent

<i>A</i>	<i>B</i>	<i>C</i>	$B \wedge (C \vee A)$	$A \rightarrow B$	$\neg(B \vee C)$
T	T	T	T <b>T</b> T <b>T</b> T	T <b>T</b> T	F <b>T</b> T <b>T</b>
T	T	F	T <b>T</b> F <b>T</b> T	T <b>T</b> T	F <b>T</b> T <b>F</b>
T	F	T	F <b>F</b> T <b>T</b> T	T <b>F</b> F	F <b>F</b> T <b>T</b>
T	F	F	F <b>F</b> F <b>T</b> T	T <b>F</b> F	T <b>F</b> F <b>F</b>
F	T	T	T <b>T</b> T <b>T</b> F	F <b>T</b> T	F <b>T</b> T <b>T</b>
F	T	F	T <b>F</b> F <b>F</b> F	F <b>T</b> T	F <b>T</b> T <b>F</b>
F	F	T	F <b>F</b> T <b>T</b> F	F <b>T</b> F	F <b>F</b> T <b>T</b>
F	F	F	F <b>F</b> F <b>F</b> F	F <b>T</b> F	T <b>F</b> F <b>F</b>

4.  $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$  Jointly consistent (see line 8)

<i>A</i>	<i>B</i>	<i>C</i>	$A \leftrightarrow (B \vee C)$	$C \rightarrow \neg A$	$A \rightarrow \neg B$
T	T	T	T <b>T</b> T <b>T</b> T	T <b>F</b> F <b>T</b>	T <b>F</b> F <b>T</b>
T	T	F	T <b>T</b> T <b>T</b> F	F <b>T</b> F <b>T</b>	T <b>F</b> F <b>T</b>
T	F	T	T <b>T</b> F <b>T</b> T	T <b>F</b> F <b>T</b>	T <b>T</b> T <b>F</b>
T	F	F	T <b>F</b> F <b>F</b> F	F <b>T</b> F <b>T</b>	T <b>T</b> T <b>F</b>
F	T	T	F <b>F</b> T <b>T</b> T	T <b>T</b> T <b>F</b>	F <b>T</b> F <b>T</b>
F	T	F	F <b>F</b> T <b>T</b> F	F <b>T</b> T <b>F</b>	F <b>T</b> F <b>T</b>
F	F	T	F <b>F</b> F <b>T</b> T	T <b>T</b> T <b>F</b>	F <b>T</b> T <b>F</b>
F	F	F	F <b>T</b> F <b>F</b> F	F <b>T</b> T <b>F</b>	F <b>T</b> T <b>F</b>

C. Use truth tables to determine whether each argument is valid or invalid.

1.  $A \rightarrow A \therefore A$  Invalid (see line 2)

<i>A</i>	$A \rightarrow A$	<i>A</i>
T	T <b>T</b> T	T
F	F <b>T</b> F	F

2.  $A \rightarrow (A \wedge \neg A) \therefore \neg A$  Valid

<i>A</i>	$A \rightarrow (A \wedge \neg A)$	$\neg A$
T	T <b>F</b> T <b>F</b> F <b>T</b>	F <b>T</b>
F	F <b>T</b> F <b>F</b> T <b>F</b>	T <b>F</b>

3.  $A \vee (B \rightarrow A) \therefore \neg A \rightarrow \neg B$  Valid

$A$	$B$	$A \vee (B \rightarrow A)$	$\neg A \rightarrow \neg B$
T	T	T T T T T	F T T F T
T	F	T T F T T	F T T T F
F	T	F F T F F	T F F F T
F	F	F T F T F	T F T T F

4.  $A \vee B, B \vee C, \neg A \therefore B \wedge C$  Invalid (see line 6)

$A$	$B$	$C$	$A \vee B$	$B \vee C$	$\neg A$	$B \wedge C$
T	T	T	T T T	T T T	F T	T T T
T	T	F	T T T	T T F	F T	T F F
T	F	T	T T F	F T T	F T	F F T
T	F	F	T T F	F F F	F T	F F F
T	T	T	F T T	T T T	T F	T T T
T	T	F	F T T	T T F	T F	T F F
T	F	T	F F F	F T T	T F	F F T
T	F	F	F F F	F F F	T F	F F F

5.  $(B \wedge A) \rightarrow C, (C \wedge A) \rightarrow B \therefore (C \wedge B) \rightarrow A$  Invalid (see line 5)

$A$	$B$	$C$	$(B \wedge A) \rightarrow C$	$(C \wedge A) \rightarrow B$	$(C \wedge B) \rightarrow A$
T	T	T	T T T T T	T T T T T	T T T T T
T	T	F	T T T F F	F F T T T	F F T T T
T	F	T	F F T T T	T T T F F	T F F T T
T	F	F	F F T T F	F F T T F	F F F T T
F	T	T	T F F T T	T F F T T	T T T F F
F	T	F	T F F T F	F F F T T	F F T T F
F	F	T	F F F T T	T F F T F	T F F T F
F	F	F	F F F T F	F F F T F	F F F T F

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence, using a complete truth table.

- 1.  $\neg B \wedge B$  Contradiction
- 2.  $\neg D \vee D$  Tautology
- 3.  $(A \wedge B) \vee (B \wedge A)$  Contingent
- 4.  $\neg[A \rightarrow (B \rightarrow A)]$  Contradiction
- 5.  $A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$  Contradiction
- 6.  $[(A \wedge B) \leftrightarrow B] \rightarrow (A \rightarrow B)$  Contingent

E. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write “equivalent.” Otherwise write, “Not equivalent.”

- 1.  $A$  and  $\neg A$
- 2.  $A \wedge \neg A$  and  $\neg B \leftrightarrow B$
- 3.  $[(A \vee B) \vee C]$  and  $[A \vee (B \vee C)]$

4.  $A \vee (B \wedge C)$  and  $(A \vee B) \wedge (A \vee C)$
5.  $[A \wedge (A \vee B)] \rightarrow B$  and  $A \rightarrow B$

**F.** Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are equivalent, write “equivalent.” Otherwise write, “not equivalent.”

1.  $A \rightarrow A$  and  $A \leftrightarrow A$
2.  $\neg(A \rightarrow B)$  and  $\neg A \rightarrow \neg B$
3.  $A \vee B$  and  $\neg A \rightarrow B$
4.  $(A \rightarrow B) \rightarrow C$  and  $A \rightarrow (B \rightarrow C)$
5.  $A \leftrightarrow (B \leftrightarrow C)$  and  $A \wedge (B \wedge C)$

**G.** Determine whether each collection of sentences is jointly consistent or jointly inconsistent using a complete truth table.

1.  $A \wedge \neg B, \neg(A \rightarrow B), B \rightarrow A$

$A$	$\wedge$	$\neg$	$B$	$\neg$	$(A$	$\rightarrow$	$B)$	$B$	$\rightarrow$	$A$	Consistent
T	F	F	T	F	T	T	T	T	T	T	
T	T	T	F	T	T	F	F	F	T	T	
F	F	F	T	F	F	T	T	T	F	F	
F	F	T	F	F	F	T	F	F	T	F	

2.  $A \vee B, A \rightarrow \neg A, B \rightarrow \neg B$

$A$	$\vee$	$B$	$A$	$\rightarrow$	$\neg$	$A$	$B$	$\rightarrow$	$\neg$	$B$	Inconsistent
T	T	T	T	F	F	T	T	F	F	T	
T	T	F	T	F	F	T	F	T	T	F	
F	T	T	F	T	T	F	T	F	F	T	
F	F	F	F	T	T	F	F	T	T	F	

3.  $\neg(\neg A \vee B), A \rightarrow \neg C, A \rightarrow (B \rightarrow C)$

Consistent

$\neg$	$(\neg$	$A$	$\vee$	$B)$	$A$	$\rightarrow$	$\neg$	$C$	$A$	$\rightarrow$	$(B$	$\rightarrow$	$C)$
F	F	T	T	T	T	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	F	T	F	T	F	F
T	F	T	F	F	T	F	F	T	T	T	F	T	T
T	F	T	F	F	T	T	T	F	T	T	F	T	F
F	T	F	T	T	F	T	F	T	F	F	T	T	T
F	T	F	T	T	F	T	T	F	F	T	T	F	F
F	T	F	T	F	F	T	F	T	F	T	F	T	T
F	T	F	T	F	F	T	T	F	F	T	F	T	F

4.  $A \rightarrow B, A \wedge \neg B$  Inconsistent  
 5.  $A \rightarrow (B \rightarrow C), (A \rightarrow B) \rightarrow C, A \rightarrow C$  Consistent

**H.** Determine whether each collection of sentences is jointly consistent or jointly inconsistent, using a complete truth table.

1.  $\neg B, A \rightarrow B, A$  Inconsistent  
 2.  $\neg(A \vee B), A \leftrightarrow B, B \rightarrow A$  Consistent  
 3.  $A \vee B, \neg B, \neg B \rightarrow \neg A$  Inconsistent  
 4.  $A \leftrightarrow B, \neg B \vee \neg A, A \rightarrow B$  Consistent  
 5.  $(A \vee B) \vee C, \neg A \vee \neg B, \neg C \vee \neg B$  Consistent

**I.** Determine whether each argument is valid or invalid, using a complete truth table.

1.  $A \rightarrow B, B \therefore A$  Invalid  
 2.  $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$  Valid  
 3.  $A \rightarrow B, A \rightarrow C \therefore B \rightarrow C$  Invalid.  
 4.  $A \rightarrow B, B \rightarrow A \therefore A \leftrightarrow B$  Valid

**J.** Determine whether each argument is valid or invalid, using a complete truth table.

1.  $A \vee [A \rightarrow (A \leftrightarrow A)] \therefore A$  Invalid  
 2.  $A \vee B, B \vee C, \neg B \therefore A \wedge C$  Valid  
 3.  $A \rightarrow B, \neg A \therefore \neg B$  Invalid  
 4.  $A, B \therefore \neg(A \rightarrow \neg B)$  Valid  
 5.  $\neg(A \wedge B), A \vee B, A \leftrightarrow B \therefore C$  Valid

**K.** Answer each of the questions below and justify your answer.

- Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are logically equivalent. What can you say about  $\mathcal{A} \leftrightarrow \mathcal{B}$ ?  
 $\mathcal{A}$  and  $\mathcal{B}$  have the same truth value on every line of a complete truth table, so  $\mathcal{A} \leftrightarrow \mathcal{B}$  is true on every line. It is a tautology.
- Suppose that  $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$  is neither a tautology nor a contradiction. What can you say about whether  $\mathcal{A}, \mathcal{B} \therefore \mathcal{C}$  is valid?  
 Since the sentence  $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$  is not a tautology, there is some line on which it is false. Since it is a conditional, on that line,  $\mathcal{A}$  and  $\mathcal{B}$  are true and  $\mathcal{C}$  is false. So the argument is invalid.
- Suppose that  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  are jointly inconsistent. What can you say about  $(\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C})$ ?  
 Since the sentences are jointly inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a contradiction
- Suppose that  $\mathcal{A}$  is a contradiction. What can you say about whether  $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$ ?

Since  $\mathcal{A}$  is false on every line of a complete truth table, there is no line on which  $\mathcal{A}$  and  $\mathcal{B}$  are true and  $\mathcal{C}$  is false. So the entailment holds.

5. Suppose that  $\mathcal{C}$  is a tautology. What can you say about whether  $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$ ? Since  $\mathcal{C}$  is true on every line of a complete truth table, there is no line on which  $\mathcal{A}$  and  $\mathcal{B}$  are true and  $\mathcal{C}$  is false. So the entailment holds.

6. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are logically equivalent. What can you say about  $(\mathcal{A} \vee \mathcal{B})$ ?

Not much. Since  $\mathcal{A}$  and  $\mathcal{B}$  are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is logically equivalent to them.

7. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are *not* logically equivalent. What can you say about  $(\mathcal{A} \vee \mathcal{B})$ ?

$\mathcal{A}$  and  $\mathcal{B}$  have different truth values on at least one line of a complete truth table, and  $(\mathcal{A} \vee \mathcal{B})$  will be true on that line. On other lines, it might be true or false. So  $(\mathcal{A} \vee \mathcal{B})$  is either a tautology or it is contingent; it is *not* a contradiction.

L. Consider the following principle:

- Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are logically equivalent. Suppose an argument contains  $\mathcal{A}$  (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced  $\mathcal{A}$  with  $\mathcal{B}$ .

Is this principle correct? Explain your answer.

The principle is correct. Since  $\mathcal{A}$  and  $\mathcal{B}$  are logically equivalent, they have the same truth table. So every valuation that makes  $\mathcal{A}$  true also makes  $\mathcal{B}$  true, and every valuation that makes  $\mathcal{A}$  false also makes  $\mathcal{B}$  false. So if no valuation makes all the premises true and the conclusion false, when  $\mathcal{A}$  was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace  $\mathcal{A}$  with  $\mathcal{B}$ .

CHAPTER 12

# *Truth table shortcuts*

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1.  $\neg B \wedge B$

Contradiction

$B$	$\neg B \wedge B$
T	F <b>F</b>
F	<b>F</b>

2.  $\neg D \vee D$

Tautology

$D$	$\neg D \vee D$
T	<b>T</b>
F	T <b>T</b>

3.  $(A \wedge B) \vee (B \wedge A)$

Neither

$A$	$B$	$(A \wedge B) \vee (B \wedge A)$
T	T	T <b>T</b>
T	F	F <b>F</b> F
F	T	F <b>F</b> F
F	F	F <b>F</b> F

4.  $\neg[A \rightarrow (B \rightarrow A)]$

Contradiction

$A$	$B$	$\neg[A \rightarrow (B \rightarrow A)]$
T	T	<b>F</b> T T
T	F	<b>F</b> T T
F	T	<b>F</b> T
F	F	<b>F</b> T

5.  $A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$

Contradiction

$A$	$B$	$A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$
T	T	<b>F</b> F FF
T	F	<b>F</b> F F
F	T	<b>F</b> T
F	F	<b>F</b> T

6.  $\neg(A \wedge B) \leftrightarrow A$

Neither

$A$	$B$	$\neg(A \wedge B) \leftrightarrow A$
T	T	F T <b>F</b>
T	F	T F <b>T</b>
F	T	T F <b>F</b>
F	F	T F <b>F</b>

7.  $A \rightarrow (B \vee C)$

Neither

$A$	$B$	$C$	$A \rightarrow (B \vee C)$
T	T	T	<b>T</b> T
T	T	F	<b>T</b> T
T	F	T	<b>T</b> T
T	F	F	<b>F</b> F
F	T	T	<b>T</b>
F	T	F	<b>T</b>
F	F	T	<b>T</b>
F	F	F	<b>T</b>

8.  $(A \wedge \neg A) \rightarrow (B \vee C)$

Tautology

$A$	$B$	$C$	$(A \wedge \neg A) \rightarrow (B \vee C)$
T	T	T	<b>FF</b> T
T	T	F	<b>FF</b> T
T	F	T	<b>FF</b> T
T	F	F	<b>FF</b> T
F	T	T	<b>F</b> T
F	T	F	<b>F</b> T
F	F	T	<b>F</b> T
F	F	F	<b>F</b> T

9.  $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$

Neither

$A$	$B$	$C$	$D$	$(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$			
T	T	T	T	T	<b>T</b>	T	T
T	T	T	F	F	<b>F</b>	T	T
T	T	F	T	T	<b>T</b>	T	T
T	T	F	F	F	<b>F</b>	T	T
T	F	T	T	F	<b>F</b>	T	T
T	F	T	F	F	<b>F</b>	T	T
T	F	F	T	F	<b>F</b>	T	T
T	F	F	F	F	<b>F</b>	T	T
F	T	T	T	T	<b>F</b>	F	T
F	T	T	F	F	<b>T</b>	F	T
F	T	F	T	T	<b>T</b>	T	F
F	T	F	F	F	<b>F</b>	T	F
F	F	T	T	F	<b>T</b>	F	T
F	F	T	F	F	<b>T</b>	F	T
F	F	F	T	F	<b>F</b>	T	F
F	F	F	F	F	<b>F</b>	T	F



CHAPTER 13

# *Partial truth tables*

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are logically equivalent:

1.  $A, \neg A$  Not logically equivalent

$A$	$A$	$\neg A$
T	T	F

2.  $A, A \vee A$  Logically equivalent

$A$	$A$	$A \vee A$
T	T	T
T	T	T

3.  $A \rightarrow A, A \leftrightarrow A$  Logically equivalent

$A$	$A \rightarrow A$	$A \leftrightarrow A$
T	T	T
F	T	T

4.  $A \vee \neg B, A \rightarrow B$  Not logically equivalent

$A$	$B$	$A \vee \neg B$	$A \rightarrow B$
T	F	T	F

5.  $A \wedge \neg A, \neg B \leftrightarrow B$  Logically equivalent

$A$	$B$	$A \wedge \neg A$	$\neg B \leftrightarrow B$
T	T	<b>FF</b>	F <b>F</b>
T	F	<b>FF</b>	T <b>F</b>
F	T	<b>F</b>	F <b>F</b>
F	F	<b>F</b>	T <b>F</b>

6.  $\neg(A \wedge B), \neg A \vee \neg B$  Logically equivalent

$A$	$B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

7.  $\neg(A \rightarrow B), \neg A \rightarrow \neg B$  Not logically equivalent

$A$	$B$	$\neg(A \rightarrow B)$	$\neg A \rightarrow \neg B$
T	T	F	F

8.  $(A \rightarrow B), (\neg B \rightarrow \neg A)$  Logically equivalent

$A$	$B$	$(A \rightarrow B)$	$(\neg B \rightarrow \neg A)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

**B.** Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly consistent, or jointly inconsistent:

1.  $A \wedge B, C \rightarrow \neg B, C$  Jointly inconsistent

$A$	$B$	$C$	$A \wedge B$	$C \rightarrow \neg B$	$C$
T	T	T	T	F	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	F	F	T
F	T	F	F	T	F
F	F	T	F	T	T
F	F	F	F	T	F

2.  $A \rightarrow B, B \rightarrow C, A, \neg C$  Jointly inconsistent

$A$	$B$	$C$	$A \rightarrow B$	$B \rightarrow C$	$A$	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

3.  $A \vee B, B \vee C, C \rightarrow \neg A$  Jointly consistent

$A$	$B$	$C$	$A \vee B$	$B \vee C$	$C \rightarrow \neg A$
F	T	T	T	T	T T

4.  $A, B, C, \neg D, \neg E, F$  Jointly consistent

$A$	$B$	$C$	$D$	$E$	$F$	$A$	$B$	$C$	$\neg D$	$\neg E$	$F$
T	T	T	F	F	T	T	T	T	T	T	T

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

1.  $A \vee [A \rightarrow (A \leftrightarrow A)] \therefore A$  Invalid

$A$	$A \vee [A \rightarrow (A \leftrightarrow A)]$	$A$
F	T T	F

2.  $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$  Invalid

$A$	$B$	$A \leftrightarrow \neg(B \leftrightarrow A)$	$A$
F	F	T F T	F

3.  $A \rightarrow B, B \therefore A$  Invalid

$A$	$B$	$A \rightarrow B$	$B$	$A$
F	T	T	T	F

4.  $A \vee B, B \vee C, \neg B \therefore A \wedge C$  Valid

$A$	$B$	$C$	$A \vee B$	$B \vee C$	$\neg B$	$A \wedge C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F	T	F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5.  $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$  Valid

$A$	$B$	$C$	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T	T	F	F
F	F	F			T

**D.** Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1.  $A \rightarrow \neg A$

Contingent

$A$	$A$	$\rightarrow$	$\neg$	$A$
$T$	$T$	<b>F</b>	$F$	$T$
$F$	$F$	<b>T</b>	$T$	$F$

2.  $A \rightarrow (A \wedge (A \vee B))$

Tautology

$A$	$B$	$A$	$\rightarrow$	$(A$	$\wedge$	$(A$	$\vee$	$B))$
$T$	$T$	$T$	<b>T</b>	$T$	$T$	$T$	$T$	$T$
$T$	$F$	$T$	<b>T</b>	$T$	$T$	$T$	$T$	$F$
$F$	$T$	$F$	<b>T</b>	$F$	$F$	$F$	$T$	$T$
$F$	$F$	$F$	<b>T</b>	$F$	$F$	$F$	$F$	$F$

3.  $(A \rightarrow B) \leftrightarrow (B \rightarrow A)$

Contingent

$A$	$B$	$(A$	$\rightarrow$	$B)$	$\leftrightarrow$	$(B$	$\rightarrow$	$A)$
$T$	$T$	$T$	$T$	<b>T</b>	$T$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	<b>F</b>	$F$	$T$	$T$	$T$
$F$	$T$	$F$	$T$	<b>F</b>	$T$	$F$	$F$	$F$
$F$	$F$	$F$	$T$	<b>T</b>	$F$	$F$	$T$	$F$

4.  $A \rightarrow \neg(A \wedge (A \vee B))$

Contingent

$A$	$B$	$A$	$\rightarrow$	$\neg$	$(A$	$\wedge$	$(A$	$\vee$	$B))$
$T$	$T$	$T$	<b>F</b>	$F$	$T$	$T$	$T$	$T$	$T$
$T$	$F$	$T$	<b>F</b>	$F$	$T$	$T$	$T$	$T$	$F$
$F$	$T$	$F$	<b>T</b>	$T$	$F$	$F$	$F$	$T$	$T$
$F$	$F$	$F$	<b>T</b>	$T$	$F$	$F$	$F$	$F$	$F$

5.  $\neg B \rightarrow [(\neg A \wedge A) \vee B]$

Contingent

$A$	$B$	$\neg$	$B$	$\rightarrow$	$((\neg$	$A$	$\wedge$	$A)$	$\vee$	$B)$
$T$	$T$	$F$	$T$	<b>T</b>	$F$	$T$	$F$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	<b>F</b>	$F$	$T$	$F$	$T$	$F$	$F$
$F$	$T$	$F$	$T$	<b>T</b>	$T$	$F$	$F$	$F$	$T$	$T$
$F$	$F$	$T$	$F$	<b>F</b>	$T$	$F$	$F$	$F$	$F$	$F$

6.  $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

Tautology

$A$	$B$	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	$A$	$\wedge$	$\neg$	$B)$
$T$	$T$	$F$	$T$	$T$	<b>T</b>	$F$	$T$	$F$	$F$	$F$	$T$
$T$	$F$	$F$	$T$	$T$	<b>T</b>	$F$	$F$	$T$	$F$	$T$	$F$
$F$	$T$	$F$	$F$	$T$	<b>T</b>	$T$	$F$	$F$	$F$	$F$	$T$
$F$	$F$	$T$	$F$	$F$	<b>T</b>	$T$	$T$	$F$	$T$	$T$	$F$

7.  $[(A \wedge B) \wedge C] \rightarrow B$

Tautology

<i>A</i>	<i>B</i>	<i>C</i>	$((A \wedge B) \wedge C)$	$\rightarrow$	<i>B</i>
<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>

8.  $\neg[(C \vee A) \vee B]$

Contingent

<i>A</i>	<i>B</i>	<i>C</i>	$\neg$	$((C \vee A) \vee B)$
<i>T</i>	<i>T</i>	<i>T</i>	<b>F</b>	<i>T</i>
<i>T</i>	<i>T</i>	<i>F</i>	<b>F</b>	<i>T</i>
<i>T</i>	<i>F</i>	<i>T</i>	<b>F</b>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<b>F</b>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<b>F</b>	<i>T</i>
<i>F</i>	<i>T</i>	<i>F</i>	<b>F</b>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<b>F</b>	<i>T</i>
<i>F</i>	<i>F</i>	<i>F</i>	<b>T</b>	<i>F</i>

9.  $[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$

Contradiction

<i>A</i>	<i>B</i>	<i>C</i>	$((A \wedge B) \wedge \neg(A \wedge B))$	$\wedge$	<i>C</i>
<i>T</i>	<i>T</i>	<i>T</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>T</i>	<i>T</i>	<i>F</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>T</i>	<i>F</i>	<i>T</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>T</i>	<i>F</i>	<i>F</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>F</i>	<i>T</i>	<i>T</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>F</i>	<i>T</i>	<i>F</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>F</i>	<i>F</i>	<i>T</i>	<b>F</b>	<i>T</i>	<b>F</b>
<i>F</i>	<i>F</i>	<i>F</i>	<b>F</b>	<i>T</i>	<b>F</b>

10.  $(A \wedge B) \rightarrow [(A \wedge C) \vee (B \wedge D)]$

Contingent

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	$((A \wedge B) \rightarrow ((A \wedge C) \vee (B \wedge D)))$
<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<b>F</b>

E. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $\neg(A \vee A)$	Contradiction
2. $(A \rightarrow B) \vee (B \rightarrow A)$	Tautology
3. $[(A \rightarrow B) \rightarrow A] \rightarrow A$	Tautology
4. $\neg[(A \rightarrow B) \vee (B \rightarrow A)]$	Contradiction
5. $(A \wedge B) \vee (A \vee B)$	Contingent
6. $\neg(A \wedge B) \leftrightarrow A$	Contingent
7. $A \rightarrow (B \vee C)$	Contingent
8. $(A \wedge \neg A) \rightarrow (B \vee C)$	Tautology
9. $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$	Contingent
10. $\neg[(A \rightarrow B) \vee (C \rightarrow D)]$	Contingent

**F.** Determine whether each the following pairs of sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write “equivalent.” Otherwise write, “not equivalent.”

1.  $A$  and  $A \vee A$
2.  $A$  and  $A \wedge A$
3.  $A \vee \neg B$  and  $A \rightarrow B$
4.  $(A \rightarrow B)$  and  $(\neg B \rightarrow \neg A)$
5.  $\neg(A \wedge B)$  and  $\neg A \vee \neg B$
6.  $((U \rightarrow (X \vee X)) \vee U)$  and  $\neg(X \wedge (X \wedge U))$
7.  $((C \wedge (N \leftrightarrow C)) \leftrightarrow C)$  and  $(\neg\neg\neg N \rightarrow C)$
8.  $[(A \vee B) \wedge C]$  and  $[A \vee (B \wedge C)]$
9.  $((L \wedge C) \wedge I)$  and  $L \vee C$

**G.** Determine whether each collection of sentences is jointly consistent or jointly inconsistent. Justify your answer with a complete or partial truth table where appropriate.

1. $A \rightarrow A, \neg A \rightarrow \neg A, A \wedge A, A \vee A$	Consistent
2. $A \rightarrow \neg A, \neg A \rightarrow A$	Inconsistent
3. $A \vee B, A \rightarrow C, B \rightarrow C$	Consistent
4. $A \vee B, A \rightarrow C, B \rightarrow C, \neg C$	Inconsistent
5. $B \wedge (C \vee A), A \rightarrow B, \neg(B \vee C)$	Inconsistent
6. $(A \leftrightarrow B) \rightarrow B, B \rightarrow \neg(A \leftrightarrow B), A \vee B$	Consistent
7. $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$	Consistent
8. $A \leftrightarrow B, \neg B \vee \neg A, A \rightarrow B$	Consistent
9. $A \leftrightarrow B, A \rightarrow C, B \rightarrow D, \neg(C \vee D)$	Consistent
10. $\neg(A \wedge \neg B), B \rightarrow \neg A, \neg B$	Consistent

**H.** Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1.  $A \rightarrow (A \wedge \neg A) \therefore \neg A$  Valid

2.  $A \vee B, A \rightarrow B, B \rightarrow A \therefore A \leftrightarrow B$  Valid
3.  $A \vee (B \rightarrow A) \therefore \neg A \rightarrow \neg B$  Valid
4.  $A \vee B, A \rightarrow B, B \rightarrow A \therefore A \wedge B$  Valid
5.  $(B \wedge A) \rightarrow C, (C \wedge A) \rightarrow B \therefore (C \wedge B) \rightarrow A$  Invalid
6.  $\neg(\neg A \vee \neg B), A \rightarrow \neg C \therefore A \rightarrow (B \rightarrow C)$  Invalid
7.  $A \wedge (B \rightarrow C), \neg C \wedge (\neg B \rightarrow \neg A) \therefore C \wedge \neg C$  Valid
8.  $A \wedge B, \neg A \rightarrow \neg C, B \rightarrow \neg D \therefore A \vee B$  Invalid
9.  $A \rightarrow B \therefore (A \wedge B) \vee (\neg A \wedge \neg B)$  Invalid
10.  $\neg A \rightarrow B, \neg B \rightarrow C, \neg C \rightarrow A \therefore \neg A \rightarrow (\neg B \vee \neg C)$  Invalid

I. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1.  $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$  Invalid
2.  $A \vee B, B \vee C, \neg A \therefore B \wedge C$  Invalid
3.  $A \rightarrow C, E \rightarrow (D \vee B), B \rightarrow \neg D \therefore (A \vee C) \vee (B \rightarrow (E \wedge D))$  Invalid
4.  $A \vee B, C \rightarrow A, C \rightarrow B \therefore A \rightarrow (B \rightarrow C)$  Invalid
5.  $A \rightarrow B, \neg B \vee A \therefore A \leftrightarrow B$  Valid

CHAPTER 15

# Basic rules for TFL

A. The following two ‘proofs’ are *incorrect*. Explain the mistakes they make.

1		$(\neg L \wedge A) \vee L$	
2		$\neg L \wedge A$	
3		$\neg L$	$\wedge E$ 3
4		$A$	$\wedge E$ 1
5		$L$	
6		$\perp$	$\neg E$ 5, 3
7		$A$	X 6
8		$A$	$\vee E$ 1, 2–4, 5–7

$\wedge E$  on line 5 can’t be applied to line 1, since it is not of the form  $\mathcal{A} \wedge \mathcal{B}$ . ‘ $A$ ’ could be obtained by  $\wedge E$ , but from line 2.

$\perp I$  on line 5 illicitly refers to a line from a closed subproof (line 3).

1		$A \wedge (B \wedge C)$	
2		$(B \vee C) \rightarrow D$	
3		$B$	$\wedge E$ 1
4		$B \vee C$	$\vee I$ 3
5		$D$	$\rightarrow E$ 4, 2

$\wedge E$  on line 3 should yield ‘ $B \wedge C$ ’. ‘ $B$ ’ could then be obtained by  $\wedge E$  again.

The citation for line 5 is the wrong way round: it should be ‘ $\rightarrow E$  2, 4’.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into *bona fide* proofs. Additionally, write down the argument that corresponds to each proof.



1	$P \wedge S$	
2	$S \rightarrow R$	
3	$P$	$\wedge E$ 1
4	$S$	$\wedge E$ 1
5	$R$	$\rightarrow E$ 2, 4
6	$R \vee E$	$\vee I$ 5

Corresponding argument:  
 $P \wedge S, S \rightarrow R \therefore R \vee E$

1	$A \rightarrow D$	
2	$A \wedge B$	
3	$A$	$\wedge E$ 2
4	$D$	$\rightarrow E$ 1, 3
5	$D \vee E$	$\vee I$ 4
6	$(A \wedge B) \rightarrow (D \vee E)$	$\rightarrow I$ 2-5

Corresponding argument:  
 $A \rightarrow D \therefore (A \wedge B) \rightarrow (D \vee E)$

1	$\neg L \rightarrow (J \vee L)$	
2	$\neg L$	
3	$J \vee L$	$\rightarrow E$ 1, 2
4	$J$	
5	$J \wedge J$	$\wedge I$ 4, 4
6	$J$	$\wedge E$ 5
7	$L$	
8	$\perp$	$\neg E$ 7, 2
9	$J$	$X$ 8
10	$J$	$\vee E$ 3, 4-6, 7-9

Corresponding argument:  
 $\neg L \rightarrow (J \vee L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1.  $J \rightarrow \neg J \therefore \neg J$

1	$J \rightarrow \neg J$	
2	$J$	
3	$\neg J$	$\rightarrow E$ 1, 2
4	$\perp$	$\neg E$ 2, 3
5	$\neg J$	$\neg I$ 2-4

2.  $Q \rightarrow (Q \wedge \neg Q) \therefore \neg Q$

1	$Q \rightarrow (Q \wedge \neg Q)$	
2	$Q$	
3	$Q \wedge \neg Q$	$\rightarrow E$ 1, 2
4	$\neg Q$	$\wedge E$ 3
5	$\perp$	$\neg E$ 2, 4
6	$\neg Q$	$\neg I$ 2-5

3.  $A \rightarrow (B \rightarrow C) \therefore (A \wedge B) \rightarrow C$

- |   |                                   |                      |
|---|-----------------------------------|----------------------|
| 1 | $A \rightarrow (B \rightarrow C)$ |                      |
| 2 | $A \wedge B$                      |                      |
| 3 | $A$                               | $\wedge E$ 2         |
| 4 | $B \rightarrow C$                 | $\rightarrow E$ 1, 3 |
| 5 | $B$                               | $\wedge E$ 2         |
| 6 | $C$                               | $\rightarrow E$ 4, 5 |
| 7 | $(A \wedge B) \rightarrow C$      | $\rightarrow I$ 2-6  |
4.  $K \wedge L \therefore K \leftrightarrow L$
- |   |                       |                              |
|---|-----------------------|------------------------------|
| 1 | $K \wedge L$          |                              |
| 2 | $K$                   |                              |
| 3 | $L$                   | $\wedge E$ 1                 |
| 4 | $L$                   |                              |
| 5 | $K$                   | $\wedge E$ 1                 |
| 6 | $K \leftrightarrow L$ | $\leftrightarrow I$ 2-3, 4-5 |
5.  $(C \wedge D) \vee E \therefore E \vee D$
- |   |                       |                      |
|---|-----------------------|----------------------|
| 1 | $(C \wedge D) \vee E$ |                      |
| 2 | $C \wedge D$          |                      |
| 3 | $D$                   | $\wedge E$ 2         |
| 4 | $E \vee D$            | $\vee I$ 3           |
| 5 | $E$                   |                      |
| 6 | $E \vee D$            | $\vee I$ 5           |
| 7 | $E \vee D$            | $\vee E$ 1, 2-4, 5-6 |
6.  $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$
- |   |                       |                              |
|---|-----------------------|------------------------------|
| 1 | $A \leftrightarrow B$ |                              |
| 2 | $B \leftrightarrow C$ |                              |
| 3 | $A$                   |                              |
| 4 | $B$                   | $\leftrightarrow E$ 1, 3     |
| 5 | $C$                   | $\leftrightarrow E$ 2, 4     |
| 6 | $C$                   |                              |
| 7 | $B$                   | $\leftrightarrow E$ 2, 6     |
| 8 | $A$                   | $\leftrightarrow E$ 1, 7     |
| 9 | $A \leftrightarrow C$ | $\leftrightarrow I$ 3-5, 6-8 |
7.  $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$

- |    |                        |                      |
|----|------------------------|----------------------|
| 1  | $\neg F \rightarrow G$ |                      |
| 2  | $F \rightarrow H$      |                      |
|    |                        |                      |
| 3  | $\neg(G \vee H)$       |                      |
|    |                        |                      |
| 4  | $F$                    |                      |
|    |                        |                      |
| 5  | $H$                    | $\rightarrow E$ 2, 4 |
|    |                        |                      |
| 6  | $G \vee H$             | $\vee I$ 5           |
|    |                        |                      |
| 7  | $\perp$                | $\neg E$ 3, 6        |
|    |                        |                      |
| 8  | $\neg F$               | $\neg I$ 4-7         |
|    |                        |                      |
| 9  | $G$                    | $\rightarrow E$ 1, 8 |
|    |                        |                      |
| 10 | $G \vee H$             | $\vee I$ 9           |
|    |                        |                      |
| 11 | $\perp$                | $\neg E$ 3, 10       |
|    |                        |                      |
| 12 | $G \vee H$             | IP 3-11              |
8.  $(Z \wedge K) \vee (K \wedge M), K \rightarrow D \therefore D$
- |   |                                  |                      |
|---|----------------------------------|----------------------|
| 1 | $(Z \wedge K) \vee (K \wedge M)$ |                      |
| 2 | $K \rightarrow D$                |                      |
|   |                                  |                      |
| 3 | $Z \wedge K$                     |                      |
|   |                                  |                      |
| 4 | $K$                              | $\wedge E$ 3         |
|   |                                  |                      |
| 5 | $K \wedge M$                     |                      |
|   |                                  |                      |
| 6 | $K$                              | $\wedge E$ 5         |
|   |                                  |                      |
| 7 | $K$                              | $\vee E$ 1, 3-4, 5-6 |
|   |                                  |                      |
| 8 | $D$                              | $\rightarrow E$ 2, 7 |
9.  $P \wedge (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$P \rightarrow \neg R$	
	$P$	
3	$P$	$\wedge E$ 1
4	$\neg R$	$\rightarrow E$ 2, 3
5	$Q \vee R$	$\wedge E$ 1
6	$Q$	
7	$Q \vee E$	$\vee I$ 6
8	$R$	
9	$\perp$	$\neg E$ 8, 4
10	$Q \vee E$	X 9
11	$Q \vee E$	$\vee E$ 5, 6-7, 8-10

10.  $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$ 

1	$S \leftrightarrow T$	
2	$S$	
3	$T$	$\leftrightarrow E$ 1, 2
4	$T \vee S$	$\vee I$ 3
5	$T \vee S$	
6	$T$	
7	$S$	$\leftrightarrow E$ 1, 6
8	$S$	
9	$S \wedge S$	$\wedge I$ 8, 8
10	$S$	$\wedge E$ 9
11	$S$	$\vee E$ 5, 6-7, 8-10
12	$S \leftrightarrow (T \vee S)$	$\leftrightarrow I$ 2-4, 5-11

11.  $\neg(P \rightarrow Q) \therefore \neg Q$ 

1	$\neg(P \rightarrow Q)$	
2	$Q$	
3	$P$	
4	$Q \wedge Q$	$\wedge I$ 2, 2
5	$Q$	$\wedge E$ 4
6	$P \rightarrow Q$	$\rightarrow I$ 3-5
7	$\perp$	$\neg E$ 6, 1
8	$\neg Q$	$\neg I$ 2-7

12.  $\neg(P \rightarrow Q) \therefore P$ 

1	$\neg(P \rightarrow Q)$	
2	$\neg P$	
3	$P$	
4	$\perp$	$\neg E$ 3, 2
5	$Q$	X 4
6	$P \rightarrow Q$	$\rightarrow I$ 3-5
7	$\perp$	$\neg E$ 6, 1
8	$P$	IP 2-7

CHAPTER 16

# *Additional rules for TFL*

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \rightarrow \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	$W$	$\wedge E$ 2
5	$\neg B$	$\rightarrow E$ 1, 4
6	$J \wedge K$	DS 3, 5
7	$K$	$\wedge E$ 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\neg L$	
4	$\neg O$	DS 2, 3
5	$L$	$\leftrightarrow E$ 1, 4
6	$\perp$	$\neg E$ 5, 3
7	$\neg\neg L$	$\neg I$ 3–6
8	$L$	DNE 7

1	$Z \rightarrow (C \wedge \neg N)$	
2	$\neg Z \rightarrow (N \wedge \neg C)$	
3	$\neg(N \vee C)$	
4	$\neg N \wedge \neg C$	DeM 3
5	$\neg N$	$\wedge E$ 4
6	$\neg C$	$\wedge E$ 4
7	$Z$	
8	$C \wedge \neg N$	$\rightarrow E$ 1, 7
9	$C$	$\wedge E$ 8
10	$\perp$	$\neg E$ 9, 6
11	$\neg Z$	$\neg I$ 7–10
12	$N \wedge \neg C$	$\rightarrow E$ 2, 11
13	$N$	$\wedge E$ 12
14	$\perp$	$\neg E$ 13, 5
15	$\neg\neg(N \vee C)$	$\neg I$ 3–14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

1.  $E \vee F, F \vee G, \neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	$E$	DS 1, 3
5	$G$	DS 2, 3
6	$E \wedge G$	$\wedge$ I 4, 5

2.  $M \vee (N \rightarrow M) \therefore \neg M \rightarrow \neg N$

1	$M \vee (N \rightarrow M)$	
2	$\neg M$	
3	$N \rightarrow M$	DS 1, 2
4	$\neg N$	MT 3, 2
5	$\neg M \rightarrow \neg N$	$\rightarrow$ I 2-4

3.  $(M \vee N) \wedge (O \vee P), N \rightarrow P, \neg P \therefore M \wedge O$

1	$(M \vee N) \wedge (O \vee P)$	
2	$N \rightarrow P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	$\wedge$ E 1
6	$M$	DS 5, 4
7	$O \vee P$	$\wedge$ E 1
8	$O$	DS 7, 3
9	$M \wedge O$	$\wedge$ I 6, 8

4.  $(X \wedge Y) \vee (X \wedge Z), \neg(X \wedge D), D \vee M \therefore M$

1	$(X \wedge Y) \vee (X \wedge Z)$							
2	$\neg(X \wedge D)$							
3	$D \vee M$							
4	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="border-bottom: 1px solid black; padding: 2px 5px;"><math>X \wedge Y</math></td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding: 2px 5px;"><math>X</math></td> <td style="padding-left: 10px;"><math>\wedge E</math> 4</td> </tr> </table>		$X \wedge Y$			$X$	$\wedge E$ 4	
	$X \wedge Y$							
	$X$	$\wedge E$ 4						
5								
6	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="border-bottom: 1px solid black; padding: 2px 5px;"><math>X \wedge Z</math></td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding: 2px 5px;"><math>X</math></td> <td style="padding-left: 10px;"><math>\wedge E</math> 6</td> </tr> </table>		$X \wedge Z$			$X$	$\wedge E$ 6	
	$X \wedge Z$							
	$X$	$\wedge E$ 6						
7								
8	$X$	$\vee E$ 1, 4-5, 6-7						
9	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="border-bottom: 1px solid black; padding: 2px 5px;"><math>D</math></td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding: 2px 5px;"><math>X \wedge D</math></td> <td style="padding-left: 10px;"><math>\wedge I</math> 8, 9</td> </tr> </table>		$D$			$X \wedge D$	$\wedge I$ 8, 9	
	$D$							
	$X \wedge D$	$\wedge I$ 8, 9						
10								
11	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding: 2px 5px;"><math>\perp</math></td> <td style="padding-left: 10px;"><math>\neg E</math> 10, 2</td> </tr> </table>		$\perp$	$\neg E$ 10, 2				
	$\perp$	$\neg E$ 10, 2						
12	$\neg D$	$\neg I$ 9-11						
13	$M$	$DS$ 3, 12						



CHAPTER 17

# *Proof-theoretic concepts*

A. Show that each of the following sentences is a theorem:

1.  $O \rightarrow O$

1			
			$O$
			$O$
			$R\ 1$
3		$O \rightarrow O$	$\rightarrow I\ 1-2$

2.  $N \vee \neg N$

1			$\neg(N \vee \neg N)$	
			$N$	
			$N \vee \neg N$	$\vee I\ 2$
4		$\perp$		$\neg E\ 1, 3$
5		$\neg N$		$\neg I\ 2-4$
6		$N \vee \neg N$		$\vee I\ 5$
7		$\perp$		$\neg E\ 1, 6$
8		$N \vee \neg N$		$IP\ 1-7$

3. $J \leftrightarrow [J \vee (L \wedge \neg L)]$				
1	$J$			
2	$J \vee (L \wedge \neg L)$	$\vee I$ 1		
3	$J \vee (L \wedge \neg L)$			
4	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>L \wedge \neg L</math></td> <td></td> </tr> </table>	$L \wedge \neg L$		
$L \wedge \neg L$				
5	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>L</math></td> <td style="padding-left: 20px;"><math>\wedge E</math> 4</td> </tr> </table>	$L$	$\wedge E$ 4	
$L$	$\wedge E$ 4			
6	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\neg L</math></td> <td style="padding-left: 20px;"><math>\wedge E</math> 4</td> </tr> </table>	$\neg L$	$\wedge E$ 4	
$\neg L$	$\wedge E$ 4			
7	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\perp</math></td> <td style="padding-left: 20px;"><math>\neg E</math> 5, 6</td> </tr> </table>	$\perp$	$\neg E$ 5, 6	
$\perp$	$\neg E$ 5, 6			
8	$\neg(L \wedge \neg L)$	$\neg I$ 4-7		
9	$J$	$DS$ 3, 8		
10	$J \leftrightarrow [J \vee (L \wedge \neg L)]$	$\leftrightarrow I$ 1-2, 3-9		
4. $((A \rightarrow B) \rightarrow A) \rightarrow A$				
1	$(A \rightarrow B) \rightarrow A$			
2	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\neg A</math></td> <td></td> </tr> </table>	$\neg A$		
$\neg A$				
3	$\neg(A \rightarrow B)$	$MT$ 1, 2		
4	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>A</math></td> <td></td> </tr> </table>	$A$		
$A$				
5	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\perp</math></td> <td style="padding-left: 20px;"><math>\neg E</math> 4, 2</td> </tr> </table>	$\perp$	$\neg E$ 4, 2	
$\perp$	$\neg E$ 4, 2			
6	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>B</math></td> <td style="padding-left: 20px;"><math>X</math> 5</td> </tr> </table>	$B$	$X$ 5	
$B$	$X$ 5			
7	$A \rightarrow B$	$\rightarrow I$ 4-6		
8	$\perp$	$\neg E$ 7, 3		
9	$A$	$IP$ 1-8		
10	$((A \rightarrow B) \rightarrow A) \rightarrow A$	$\rightarrow I$ 1-9		

**B.** Provide proofs to show each of the following:

1.  $C \rightarrow (E \wedge G), \neg C \rightarrow G \vdash G$

1	$C \rightarrow (E \wedge G)$	
2	$\neg C \rightarrow G$	
3	<div style="border-bottom: 1px solid black; padding-bottom: 5px;"><math>\neg G</math></div>	
4	<div style="border-left: 1px solid black; padding-left: 5px;"> <div style="border-bottom: 1px solid black; padding-bottom: 5px;"><math>C</math></div> </div>	
5	<div style="border-left: 1px solid black; padding-left: 10px;"><math>E \wedge G</math></div>	$\rightarrow E$ 1, 4
6	<div style="border-left: 1px solid black; padding-left: 10px;"><math>G</math></div>	$\wedge E$ 5
7	<div style="border-left: 1px solid black; padding-left: 10px;"><math>\perp</math></div>	$\neg E$ 3, 6
8	$\neg C$	$\neg I$ 4-7
9	$G$	$\rightarrow E$ 2, 8
10	$\perp$	$\neg E$ 3, 9
11	$G$	IP 3-10
2. $M \wedge (\neg N \rightarrow \neg M) \vdash (N \wedge M) \vee \neg M$		
1	$M \wedge (\neg N \rightarrow \neg M)$	
2	$M$	$\wedge E$ 1
3	$\neg N \rightarrow \neg M$	$\wedge E$ 1
4	<div style="border-left: 1px solid black; padding-left: 5px;"> <div style="border-bottom: 1px solid black; padding-bottom: 5px;"><math>\neg N</math></div> </div>	
5	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-bottom: 1px solid black; padding-bottom: 5px;"><math>\neg M</math></div> </div>	$\rightarrow E$ 3, 4
6	<div style="border-left: 1px solid black; padding-left: 10px;"><math>\perp</math></div>	$\neg E$ 2, 5
7	$N$	IP 4-6
8	$N \wedge M$	$\wedge I$ 7, 2
9	$(N \wedge M) \vee \neg M$	$\vee I$ 8
3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \rightarrow M) \vdash Y \rightarrow Z$		
1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$D \wedge (D \rightarrow M)$	
3	$D$	$\wedge E$ 2
4	$D \rightarrow M$	$\wedge E$ 2
5	$M$	$\rightarrow E$ 4, 3
6	<div style="border-left: 1px solid black; padding-left: 5px;"> <div style="border-bottom: 1px solid black; padding-bottom: 5px;"><math>Y</math></div> </div>	
7	<div style="border-left: 1px solid black; padding-left: 10px;"><math>Y \wedge M</math></div>	$\wedge I$ 6, 5
8	$Z \wedge K$	$\leftrightarrow E$ 1, 7
9	$Z$	$\wedge E$ 8
10	$Y \rightarrow Z$	$\rightarrow I$ 6-9

4.  $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\neg Z$	
4	$W \vee X$	
5	$W$	
6	$W \vee Y$	$\vee I$ 5
7	$X$	
8	$Y$	$\rightarrow E$ 2, 7
9	$W \vee Y$	$\vee I$ 8
10	$W \vee Y$	$\vee E$ 4, 5-6, 7-9
11	$Y \vee Z$	
12	$Y$	$DS$ 11, 3
13	$W \vee Y$	$\vee I$ 12
14	$W \vee Y$	$\vee E$ 1, 4-10, 11-13

C. Show that each of the following pairs of sentences are provably equivalent:

1.  $R \leftrightarrow E, E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	$E$	
3	$R$	$\leftrightarrow E$ 1, 2
4	$R$	
5	$E$	$\leftrightarrow E$ 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow I$ 2-3, 4-5

1	$E \leftrightarrow R$	
2	$E$	
3	$R$	$\leftrightarrow E$ 1, 2
4	$R$	
5	$E$	$\leftrightarrow E$ 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow I$ 4-5, 2-3

2.  $G, \neg\neg\neg\neg G$

1	$G$	
2	$\neg\neg\neg G$	
3	$\neg G$	$DNE$ 2
4	$\perp$	$\neg E$ 1, 3
5	$\neg\neg\neg\neg G$	$\neg I$ 2-4

1	$\neg\neg\neg\neg G$	
2	$\neg\neg G$	$DNE$ 1
3	$G$	$DNE$ 2

3.  $T \rightarrow S, \neg S \rightarrow \neg T$ 

1	$T \rightarrow S$	
2	$\neg S$	
3	$\neg T$	MT 1, 2
4	$\neg S \rightarrow \neg T$	$\rightarrow$ I 2-3

1	$\neg S \rightarrow \neg T$	
2	$T$	
3	$\neg S$	
4	$\neg T$	$\rightarrow$ E 1, 3
5	$\perp$	$\neg$ E 2, 4
6	$\neg\neg S$	$\neg$ I 3-5
7	$S$	DNE 6
8	$T \rightarrow S$	$\rightarrow$ I 2-7

4.  $U \rightarrow I, \neg(U \wedge \neg I)$ 

1	$U \rightarrow I$	
2	$U \wedge \neg I$	
3	$U$	$\wedge$ E 2
4	$\neg I$	$\wedge$ E 2
5	$I$	$\rightarrow$ E 1, 3
6	$\perp$	$\neg$ E 5, 4
7	$\neg(U \wedge \neg I)$	$\neg$ I 2-6

1	$\neg(U \wedge \neg I)$	
2	$U$	
3	$\neg I$	
4	$U \wedge \neg I$	$\wedge$ I 2, 3
5	$\perp$	$\neg$ E 4, 1
6	$\neg\neg I$	$\neg$ I 3-5
7	$I$	DNE 6
8	$U \rightarrow I$	$\rightarrow$ I 2-7

5.  $\neg(C \rightarrow D), C \wedge \neg D$

1	$C \wedge \neg D$	
2	$C$	$\wedge E$ 1
3	$\neg D$	$\wedge E$ 1
4	$C \rightarrow D$	
5	$D$	$\rightarrow E$ 4, 2
6	$\perp$	$\neg E$ 5, 3
7	$\neg(C \rightarrow D)$	$\neg I$ 4-6

1	$\neg(C \rightarrow D)$	
2	$D$	
3	$C$	
4	$D$	$R$ 2
5	$C \rightarrow D$	$\rightarrow I$ 3-4
6	$\perp$	$\neg E$ 5, 1
7	$\neg D$	$\neg I$ 2-6
8	$\neg C$	
9	$C$	
10	$\perp$	$\neg E$ 9, 8
11	$D$	$X$ 10
12	$C \rightarrow D$	$\rightarrow I$ 9-11
13	$\perp$	$\neg E$ 12, 1
14	$\neg\neg C$	$\neg I$ 8-13
15	$C$	$DNE$ 14
16	$C \wedge \neg D$	$\wedge I$ 15, 7

6.  $\neg G \leftrightarrow H, \neg(G \leftrightarrow H)$

1	$\neg G \leftrightarrow H$	
2	$G \leftrightarrow H$	
3	$G$	
4	$H$	$\leftrightarrow E$ 2, 3
5	$\neg G$	$\leftrightarrow E$ 1, 4
6	$\perp$	$\neg E$ 3, 5
7	$\neg G$	
8	$H$	$\leftrightarrow E$ 1, 7
9	$G$	$\leftrightarrow E$ 2, 8
10	$\perp$	$\neg E$ 9, 7
11	$\perp$	$LEM$ 3-6, 7-10
12	$\neg(G \leftrightarrow H)$	$\neg I$ 2-11

1	$\neg(G \leftrightarrow H)$	
2	$\neg G$	
3	$\neg H$	
4	$G$	
5	$\perp$	$\neg E$ 4, 2
6	$H$	X 5
7	$H$	
8	$\perp$	$\neg E$ 7, 3
9	$G$	X 8
10	$G \leftrightarrow H$	$\leftrightarrow I$ 4-6, 7-9
11	$\perp$	$\neg E$ 10, 1
12	$H$	IP 3-11
13	$H$	
14	$G$	
15	$G$	
16	$H$	R 13
17	$H$	
18	$G$	R 14
19	$G \leftrightarrow H$	$\leftrightarrow I$ 15-16, 17-18
20	$\perp$	$\neg E$ 19, 1
21	$\neg G$	$\neg I$ 14-20
22	$\neg G \leftrightarrow H$	$\leftrightarrow I$ 2-12, 13-21

**D.** If you know that  $\mathcal{A} \vdash \mathcal{B}$ , what can you say about  $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$ ? What about  $(\mathcal{A} \vee \mathcal{C}) \vdash \mathcal{B}$ ? Explain your answers.

If  $\mathcal{A} \vdash \mathcal{B}$ , then  $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$ . After all, if  $\mathcal{A} \vdash \mathcal{B}$ , then there is some proof with assumption  $\mathcal{A}$  that ends with  $\mathcal{B}$ , and no undischarged assumptions other than  $\mathcal{A}$ . Now, if we start a proof with assumption  $(\mathcal{A} \wedge \mathcal{C})$ , we can obtain  $\mathcal{A}$  by  $\wedge E$ . We can now copy and paste the original proof of  $\mathcal{B}$  from  $\mathcal{A}$ , adding 1 to every line number and line number citation. The result will be a proof of  $\mathcal{B}$  from assumption  $\mathcal{A}$ .

However, we cannot prove much from  $(\mathcal{A} \vee \mathcal{C})$ . After all, it might be impossible to prove  $\mathcal{B}$  from  $\mathcal{C}$ .

**E.** In this chapter, we claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem.

Why did we claim this? (*Hint*: think of a sentence that would be a theorem iff  $\mathcal{A}$  and  $\mathcal{B}$  were provably equivalent.)

Consider the sentence  $\mathcal{A} \leftrightarrow \mathcal{B}$ . Suppose we can show that this is a theorem. So we can prove it, with no assumptions, in  $m$  lines, say. Then if we assume  $\mathcal{A}$  and copy and paste the proof of  $\mathcal{A} \leftrightarrow \mathcal{B}$  (changing the line numbering), we will have a deduction of this shape:

1			$\mathcal{A}$	
$m + 1$			$\mathcal{A} \leftrightarrow \mathcal{B}$	
$m + 2$			$\mathcal{B}$	$\leftrightarrow E\ m + 1, 1$

This will show that  $\mathcal{A} \vdash \mathcal{B}$ . In exactly the same way, we can show that  $\mathcal{B} \vdash \mathcal{A}$ . So if we can show that  $\mathcal{A} \leftrightarrow \mathcal{B}$  is a theorem, we can show that  $\mathcal{A}$  and  $\mathcal{B}$  are provably equivalent.

Conversely, suppose we can show that  $\mathcal{A}$  and  $\mathcal{B}$  are provably equivalent. Then we can prove  $\mathcal{B}$  from the assumption of  $\mathcal{A}$  in  $m$  lines, say, and prove  $\mathcal{A}$  from the assumption of  $\mathcal{B}$  in  $n$  lines, say. Copying and pasting these proofs together (changing the line numbering where appropriate), we obtain:

1			$\mathcal{A}$	
$m$			$\mathcal{B}$	
$m + 1$			$\mathcal{B}$	
$m + n$			$\mathcal{A}$	
$m + n + 1$			$\mathcal{A} \leftrightarrow \mathcal{B}$	$\leftrightarrow I\ 1-m, m + 1-m + n$

Thus showing that  $\mathcal{A} \leftrightarrow \mathcal{B}$  is a theorem.

There was nothing special about  $\mathcal{A}$  and  $\mathcal{B}$  in this. So what this shows is that the problem of showing that two sentences are provably equivalent is, essentially, the same problem as showing that a certain kind of sentence (a biconditional) is a theorem.



CHAPTER 19

# *Derived rules*

A. Provide proof schemes that justify the addition of the third and fourth De Morgan rules as derived rules.

Third rule:

$m$	$\neg \mathcal{A} \wedge \neg \mathcal{B}$	
$k$	$\neg \mathcal{A}$	$\wedge E\ m$
$k+1$	$\neg \mathcal{B}$	$\wedge E\ m$
$k+2$	$\mathcal{A} \vee \mathcal{B}$	
$k+3$	$\mathcal{A}$	
$k+4$	$\perp$	$\neg E\ k+3, k$
$k+5$	$\mathcal{B}$	
$k+6$	$\perp$	$\neg E\ k+5, k+1$
$k+7$	$\perp$	$\vee E\ k+2, k+3-k+4, k+5-k+6$
$k+8$	$\neg(\mathcal{A} \vee \mathcal{B})$	$\neg I\ k+2-k+7$

Fourth rule:

$m$	$\neg(A \vee B)$					
$k$	<table style="border-collapse: collapse; margin-left: 0.5em;"> <tr> <td style="border-right: 1px solid black; padding-right: 0.5em;"><math>A</math></td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 0.5em;"><math>\hline A \vee B</math></td> <td style="padding-left: 0.5em;"><math>\vee I\ k</math></td> </tr> </table>	$A$		$\hline A \vee B$	$\vee I\ k$	
$A$						
$\hline A \vee B$	$\vee I\ k$					
$k+1$	$A \vee B$					
$k+2$	$\perp$	$\neg E\ k+1, m$				
$k+3$	$\neg A$	$\neg I\ k-k+2$				
$k+4$	<table style="border-collapse: collapse; margin-left: 0.5em;"> <tr> <td style="border-right: 1px solid black; padding-right: 0.5em;"><math>B</math></td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 0.5em;"><math>\hline A \vee B</math></td> <td style="padding-left: 0.5em;"><math>\vee I\ k+4</math></td> </tr> </table>	$B$		$\hline A \vee B$	$\vee I\ k+4$	
$B$						
$\hline A \vee B$	$\vee I\ k+4$					
$k+5$	$A \vee B$					
$k+6$	$\perp$	$\neg E\ k+5, m$				
$k+7$	$\neg B$	$\neg I\ k+4-k+6$				
$k+8$	$\neg A \wedge \neg B$	$\wedge I\ k+3, k+7$				

**B.** The proofs you offered in response to the practice exercises of §§16–17 used derived rules. Replace the use of derived rules, in such proofs, with only basic rules. You will find some ‘repetition’ in the resulting proofs; in such cases, offer a streamlined proof using only basic rules. (This will give you a sense, both of the power of derived rules, and of how all the rules interact.)

## CHAPTER 20

# *Soundness and Completeness*

### Practice exercises

**A.** Use either a derivation or a truth table for each of the following.

1. Show that  $A \rightarrow [(B \wedge C) \vee D] \rightarrow A$  is a tautology.
2. Show that  $A \rightarrow (A \rightarrow B)$  is not a tautology
3. Show that the sentence  $A \rightarrow \neg A$  is not a contradiction.
4. Show that the sentence  $A \leftrightarrow \neg A$  is a contradiction.
5. Show that the sentence  $\neg(W \rightarrow (J \vee J))$  is contingent
6. Show that the sentence  $\neg(X \vee (Y \vee Z)) \vee (X \vee (Y \vee Z))$  is not contingent
7. Show that the sentence  $B \rightarrow \neg S$  is equivalent to the sentence  $\neg\neg B \rightarrow \neg S$
8. Show that the sentence  $\neg(X \vee O)$  is not equivalent to the sentence  $X \wedge O$
9. Show that the sentences  $\neg(A \vee B)$ ,  $C$ ,  $C \rightarrow A$  are jointly inconsistent.
10. Show that the sentences  $\neg(A \vee B)$ ,  $\neg B$ ,  $B \rightarrow A$  are jointly consistent
11. Show that  $\neg(A \vee (B \vee C)) \therefore \neg C$  is valid.
12. Show that  $\neg(A \wedge (B \vee C)) \therefore \neg C$  is invalid.

**B.** Use either a derivation or a truth table for each of the following.

1. Show that  $A \rightarrow (B \rightarrow A)$  is a tautology
2. Show that  $\neg(((N \leftrightarrow Q) \vee Q) \vee N)$  is not a tautology

3. Show that  $Z \vee (\neg Z \leftrightarrow Z)$  is contingent
4. show that  $(L \leftrightarrow ((N \rightarrow N) \rightarrow L)) \vee H$  is not contingent
5. Show that  $(A \leftrightarrow A) \wedge (B \wedge \neg B)$  is a contradiction
6. Show that  $(B \leftrightarrow (C \vee B))$  is not a contradiction.
7. Show that  $((\neg X \leftrightarrow X) \vee X)$  is equivalent to  $X$
8. Show that  $F \wedge (K \wedge R)$  is not equivalent to  $(F \leftrightarrow (K \leftrightarrow R))$
9. Show that the sentences  $\neg(W \rightarrow W)$ ,  $(W \leftrightarrow W) \wedge W$ ,  $E \vee (W \rightarrow \neg(E \wedge W))$  are inconsistent.
10. Show that the sentences  $\neg R \vee C$ ,  $(C \wedge R) \rightarrow \neg R$ ,  $(\neg(R \vee R) \rightarrow R)$  are consistent.
11. Show that  $\neg\neg(C \leftrightarrow \neg C)$ ,  $((G \vee C) \vee G) \therefore ((G \rightarrow C) \wedge G)$  is valid.
12. Show that  $\neg\neg L$ ,  $(C \rightarrow \neg L) \rightarrow C) \therefore \neg C$  is invalid.

## CHAPTER 22

# *Sentences with one quantifier*

A. Here are the syllogistic figures identified by Aristotle and his successors, along with their medieval names:

- **Barbara.** All G are F. All H are G. So: All H are F  
 $\forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow Fx)$
- **Celarent.** No G are F. All H are G. So: No H are F  
 $\forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Ferio.** No G are F. Some H is G. So: Some H is not F  
 $\forall x(Gx \rightarrow \neg Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Darii.** All G are H. Some H is G. So: Some H is F.  
 $\forall x(Gx \rightarrow Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge Fx)$
- **Camestres.** All F are G. No H are G. So: No H are F.  
 $\forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Cesare.** No F are G. All H are G. So: No H are F.  
 $\forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Baroko.** All F are G. Some H is not G. So: Some H is not F.  
 $\forall x(Fx \rightarrow Gx), \exists x(Hx \wedge \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Festino.** No F are G. Some H are G. So: Some H is not F.  
 $\forall x(Fx \rightarrow \neg Gx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Datisi.** All G are F. Some G is H. So: Some H is F.  
 $\forall x(Gx \rightarrow Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge Fx)$
- **Disamis.** Some G is F. All G are H. So: Some H is F.  
 $\exists x(Gx \wedge Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Ferison.** No G are F. Some G is H. So: Some H is not F.  
 $\forall x(Gx \rightarrow \neg Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Bokardo.** Some G is not F. All G are H. So: Some H is not F.  
 $\exists x(Gx \wedge \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Camenes.** All F are G. No G are H So: No H is F.  
 $\forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \forall x(Hx \rightarrow \neg Fx)$

- **Dimaris.** Some F is G. All G are H. So: Some H is F.  
 $\exists x(Fx \wedge Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Fresison.** No F are G. Some G is H. So: Some H is not F.  
 $\forall x(Fx \rightarrow \neg Gx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$

Symbolize each argument in FOL.

**B.** Using the following symbolization key:

domain: people

$Kx$ : \_\_\_\_\_<sub>x</sub> knows the combination to the safe

$Sx$ : \_\_\_\_\_<sub>x</sub> is a spy

$Vx$ : \_\_\_\_\_<sub>x</sub> is a vegetarian

$h$ : Hofthor

$i$ : Ingmar

symbolize the following sentences in FOL:

1. Neither Hofthor nor Ingmar is a vegetarian.  
 $\neg Vh \wedge \neg Vi$
2. No spy knows the combination to the safe.  
 $\forall x(Sx \rightarrow \neg Kx)$
3. No one knows the combination to the safe unless Ingmar does.  
 $\forall x \neg Kx \vee Ki$
4. Hofthor is a spy, but no vegetarian is a spy.  
 $Sh \wedge \forall x(Vx \rightarrow \neg Sx)$

**C.** Using this symbolization key:

domain: all animals

$Ax$ : \_\_\_\_\_<sub>x</sub> is an alligator.

$Mx$ : \_\_\_\_\_<sub>x</sub> is a monkey.

$Rx$ : \_\_\_\_\_<sub>x</sub> is a reptile.

$Zx$ : \_\_\_\_\_<sub>x</sub> lives at the zoo.

$a$ : Amos

$b$ : Bouncer

$c$ : Cleo

symbolize each of the following sentences in FOL:

1. Amos, Bouncer, and Cleo all live at the zoo.  
 $Za \wedge Zb \wedge Zc$
2. Bouncer is a reptile, but not an alligator.  
 $Rb \wedge \neg Ab$
3. Some reptile lives at the zoo.  
 $\exists x(Rx \wedge Zx)$
4. Every alligator is a reptile.  
 $\forall x(Ax \rightarrow Rx)$
5. Any animal that lives at the zoo is either a monkey or an alligator.

$$\forall x(Zx \rightarrow (Mx \vee Ax))$$

6. There are reptiles which are not alligators.

$$\exists x(Rx \wedge \neg Ax)$$

7. If any animal is an reptile, then Amos is.

$$\exists xRx \rightarrow Ra$$

8. If any animal is an alligator, then it is a reptile.

$$\forall x(Ax \rightarrow Rx)$$

**D.** For each argument, write a symbolization key and symbolize the argument in FOL.

1. Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat

domain: people

$Lx$ :  $\text{\_\_\_\_\_\_}x$  is a logician

$Hx$ :  $\text{\_\_\_\_\_\_}x$  wears a funny hat

$i$ : Willard

$$Li, \forall x(Lx \rightarrow Hx) \therefore Hi$$

2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.

domain: physical things

$Dx$ :  $\text{\_\_\_\_\_\_}x$  is on my desk

$Ex$ :  $\text{\_\_\_\_\_\_}x$  escapes my attention

$Cx$ :  $\text{\_\_\_\_\_\_}x$  is a computer

$$\forall x(Dx \rightarrow \neg Ex), \exists x(Dx \wedge Cx) \therefore \exists x(Cx \wedge \neg Ex)$$

3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.

domain: episodes (psychological and televised)

$Dx$ :  $\text{\_\_\_\_\_\_}x$  is one of my dreams

$Bx$ :  $\text{\_\_\_\_\_\_}x$  is in black and white

$Ox$ :  $\text{\_\_\_\_\_\_}x$  is an old TV show

$$\forall x(Dx \rightarrow Bx), \forall x(Ox \rightarrow Bx) \therefore \exists x(Dx \wedge Ox).$$

Comment: generic statements are tricky to deal with. Does the second sentence mean that *all* old TV shows are in black and white; or that most of them are; or that most of the things which are in black and white are old TV shows? I have gone with the former, but it is not clear that FOL deals with these well.

4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.

domain: people

$Ax$ :  $\text{\_\_\_\_\_\_}x$  has been to Australia

$Kx$ :  $\text{\_\_\_\_\_\_}x$  has seen a kangaroo

$Zx$ : \_\_\_\_\_ $_x$  has been to a zoo

$h$ : Holmes

$a$ : Watson

$\neg Ah \wedge \neg Aa, \forall x(Kx \rightarrow (Ax \vee Zx)), \neg Ka \wedge Kh \therefore Zh$

5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

domain: people

$Sx$ : \_\_\_\_\_ $_x$  expects the Spanish Inquisition

$Tx$ : \_\_\_\_\_ $_x$  knows the troubles I've seen

$h$ : Holmes

$a$ : Watson

$\forall x\neg Sx, \forall x\neg Tx \therefore \forall x(Sx \rightarrow Tx)$

6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

domain: people

$Bx$ : \_\_\_\_\_ $_x$  is a baby

$Ix$ : \_\_\_\_\_ $_x$  is illogical

$Cx$ : \_\_\_\_\_ $_x$  can manage a crocodile

$b$ : Berthold

$\forall x(Bx \rightarrow Ix), \forall x(Ix \rightarrow \neg Cx), Bb \therefore \neg Cb$



## CHAPTER 23

# *Multiple generality*

A. Using this symbolization key:

domain: all animals

$Ax$ : \_\_\_\_\_ $x$  is an alligator

$Mx$ : \_\_\_\_\_ $x$  is a monkey

$Rx$ : \_\_\_\_\_ $x$  is a reptile

$Zx$ : \_\_\_\_\_ $x$  lives at the zoo

$Lxy$ : \_\_\_\_\_ $x$  loves \_\_\_\_\_ $y$

$a$ : Amos

$b$ : Bouncer

$c$ : Cleo

symbolize each of the following sentences in FOL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.  
 $Lcb \rightarrow Mb$
2. If both Bouncer and Cleo are alligators, then Amos loves them both.  
 $(Ab \wedge Ac) \rightarrow (Lab \wedge Lac)$
3. Cleo loves a reptile.  
 $\exists x(Rx \wedge Lcx)$   
Comment: this English expression is ambiguous; in some contexts, it can be read as a generic, along the lines of ‘Cleo loves reptiles’. (Compare ‘I do love a good pint’.)
4. Bouncer loves all the monkeys that live at the zoo.  
 $\forall x((Mx \wedge Zx) \rightarrow Lbx)$
5. All the monkeys that Amos loves love him back.  
 $\forall x((Mx \wedge Lax) \rightarrow Lxa)$
6. Every monkey that Cleo loves is also loved by Amos.  
 $\forall x((Mx \wedge Lcx) \rightarrow Lax)$

7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.

$$\exists x(Mx \wedge Lxb \wedge \neg Lbx)$$

**B.** Using the following symbolization key:

domain: all animals

$Dx$ : \_\_\_\_\_ $x$  is a dog

$Sx$ : \_\_\_\_\_ $x$  likes samurai movies

$Lxy$ : \_\_\_\_\_ $x$  is larger than \_\_\_\_\_ $y$

$r$ : Rave

$h$ : Shane

$d$ : Daisy

symbolize the following sentences in FOL:

1. Rave is a dog who likes samurai movies.

$$Dr \wedge Sr$$

2. Rave, Shane, and Daisy are all dogs.

$$Dr \wedge Dh \wedge Dd$$

3. Shane is larger than Rave, and Daisy is larger than Shane.

$$Lhr \wedge Ldh$$

4. All dogs like samurai movies.

$$\forall x(Dx \rightarrow Sx)$$

5. Only dogs like samurai movies.

$$\forall x(Sx \rightarrow Dx)$$

*Comment: the FOL sentence just written does not require that anyone likes samurai movies. The English sentence might suggest that at least some dogs *do* like samurai movies?*

6. There is a dog that is larger than Shane.

$$\exists x(Dx \wedge Lxh)$$

7. If there is a dog larger than Daisy, then there is a dog larger than Shane.

$$\exists x(Dx \wedge Lxd) \rightarrow \exists x(Dx \wedge Lxh)$$

8. No animal that likes samurai movies is larger than Shane.

$$\forall x(Sx \rightarrow \neg Lxh)$$

9. No dog is larger than Daisy.

$$\forall x(Dx \rightarrow \neg Lxd)$$

10. Any animal that dislikes samurai movies is larger than Rave.

$$\forall x(\neg Sx \rightarrow Lxr)$$

*Comment: this is very poor, though! For 'dislikes' does not mean the same as 'does not like'.*

11. There is an animal that is between Rave and Shane in size.

$$\exists x((Lbx \wedge Lxh) \vee (Lhx \wedge Lxr))$$

12. There is no dog that is between Rave and Shane in size.

$$\forall x(Dx \rightarrow \neg[(Lbx \wedge Lxh) \vee (Lhx \wedge Lxr)])$$

13. No dog is larger than itself.

$$\forall x(Dx \rightarrow \neg Lxx)$$

14. Every dog is larger than some dog.

$$\forall x(Dx \rightarrow \exists y(Dy \wedge Lxy))$$

Comment: the English sentence is potentially ambiguous here. I have resolved the ambiguity by assuming it should be paraphrased by 'for every dog, there is a dog smaller than it'.

15. There is an animal that is smaller than every dog.

$$\exists x\forall y(Dy \rightarrow Lxy)$$

16. If there is an animal that is larger than any dog, then that animal does not like samurai movies.

$$\forall x(\forall y(Dy \rightarrow Lxy) \rightarrow \neg Sx)$$

Comment: I have assumed that 'larger than any dog' here means 'larger than every dog'.

C. Using the symbolization key given, translate each English-language sentence into FOL.

domain: candies

$Cx$ : \_\_\_\_\_<sub>x</sub> has chocolate in it.

$Mx$ : \_\_\_\_\_<sub>x</sub> has marzipan in it.

$Sx$ : \_\_\_\_\_<sub>x</sub> has sugar in it.

$Tx$ : Boris has tried \_\_\_\_\_<sub>x</sub>.

$Bxy$ : \_\_\_\_\_<sub>x</sub> is better than \_\_\_\_\_<sub>y</sub>.

1. Boris has never tried any candy.
2. Marzipan is always made with sugar.
3. Some candy is sugar-free.
4. The very best candy is chocolate.
5. No candy is better than itself.
6. Boris has never tried sugar-free chocolate.
7. Boris has tried marzipan and chocolate, but never together.
8. Any candy with chocolate is better than any candy without it.
9. Any candy with chocolate and marzipan is better than any candy that lacks both.

D. Using the following symbolization key:

domain: people and dishes at a potluck

$Rx$ : \_\_\_\_\_<sub>x</sub> has run out.

$Tx$ : \_\_\_\_\_<sub>x</sub> is on the table.

$Fx$ : \_\_\_\_\_<sub>x</sub> is food.

$Px$ : \_\_\_\_\_<sub>x</sub> is a person.

$Lxy$ : \_\_\_\_\_<sub>x</sub> likes \_\_\_\_\_<sub>y</sub>.

$e$ : Eli

$f$ : Francesca

$g$ : the guacamole

symbolize the following English sentences in FOL:

1. All the food is on the table.  
 $\forall x(Fx \rightarrow Tx)$
2. If the guacamole has not run out, then it is on the table.  
 $\neg Rg \rightarrow Tg$
3. Everyone likes the guacamole.  
 $\forall x(Px \rightarrow Lxg)$
4. If anyone likes the guacamole, then Eli does.  
 $\exists x(Px \wedge Lxg) \rightarrow Leg$
5. Francesca only likes the dishes that have run out.  
 $\forall x[(Lfx \wedge Fx) \rightarrow Rx]$
6. Francesca likes no one, and no one likes Francesca.  
 $\forall x[Px \rightarrow (\neg Lfx \wedge \neg Lxf)]$
7. Eli likes anyone who likes the guacamole.  
 $\forall x((Px \wedge Lxg) \rightarrow Lex)$
8. Eli likes anyone who likes the people that he likes.  
 $\forall x[(Px \wedge \forall y[(Py \wedge Ley) \rightarrow Lxy]) \rightarrow Lex]$
9. If there is a person on the table already, then all of the food must have run out.  
 $\exists x(Px \wedge Tx) \rightarrow \forall x(Fx \rightarrow Rx)$

E. Using the following symbolization key:

domain: people

- $Dx$ : \_\_\_\_\_ $x$  dances ballet.  
 $Fx$ : \_\_\_\_\_ $x$  is female.  
 $Mx$ : \_\_\_\_\_ $x$  is male.  
 $Cxy$ : \_\_\_\_\_ $x$  is a child of \_\_\_\_\_ $y$ .  
 $Sxy$ : \_\_\_\_\_ $x$  is a sibling of \_\_\_\_\_ $y$ .  
 $e$ : Elmer  
 $j$ : Jane  
 $p$ : Patrick

symbolize the following sentences in FOL:

1. All of Patrick's children are ballet dancers.  
 $\forall x(Cxp \rightarrow Dx)$
2. Jane is Patrick's daughter.  
 $Cjp \wedge Fj$
3. Patrick has a daughter.  
 $\exists x(Cxp \wedge Fx)$
4. Jane is an only child.  
 $\neg \exists xSxj$
5. All of Patrick's sons dance ballet.  
 $\forall x[(Cxp \wedge Mx) \rightarrow Dx]$
6. Patrick has no sons.  
 $\neg \exists x(Cxp \wedge Mx)$
7. Jane is Elmer's niece.

$$\exists x(Sxe \wedge Cjx \wedge Fj)$$

8. Patrick is Elmer's brother.

$$Spe \wedge Mp$$

9. Patrick's brothers have no children.

$$\forall x[(Sp x \wedge Mx) \rightarrow \neg \exists y Cyx]$$

10. Jane is an aunt.

$$Fj \wedge \exists x(Sxj \wedge \exists y Cyx)$$

11. Everyone who dances ballet has a brother who also dances ballet.

$$\forall x[Dx \rightarrow \exists y(My \wedge Syx \wedge Dy)]$$

12. Every woman who dances ballet is the child of someone who dances ballet.

$$\forall x[(Fx \wedge Dx) \rightarrow \exists y(Cxy \wedge Dy)]$$

## CHAPTER 24

# *Identity*

A. Explain why:

- ‘ $\exists x\forall y(Ay \leftrightarrow x = y)$ ’ is a good symbolization of ‘there is exactly one apple’.  
We might naturally read this in English thus:

- There is something,  $x$ , such that, if you choose any object at all, if you chose an apple then you chose  $x$  itself, and if you chose  $x$  itself then you chose an apple.

The  $x$  in question must therefore be the one and only thing which is an apple.

- ‘ $\exists x\exists y[\neg x = y \wedge \forall z(Az \leftrightarrow (x = z \vee y = z))]$ ’ is a good symbolization of ‘there are exactly two apples’.

Similarly to the above, we might naturally read this in English thus:

- There are two distinct things,  $x$  and  $y$ , such that if you choose any object at all, if you chose an apple then you either chose  $x$  or  $y$ , and if you chose either  $x$  or  $y$  then you chose an apple.

The  $x$  and  $y$  in question must therefore be the only things which are apples, and since they are distinct, there are two of them.

## CHAPTER 25

# *Definite descriptions*

A. Using the following symbolization key:

domain: people

$Kx$ : \_\_\_\_\_ $x$  knows the combination to the safe.

$Sx$ : \_\_\_\_\_ $x$  is a spy.

$Vx$ : \_\_\_\_\_ $x$  is a vegetarian.

$Txy$ : \_\_\_\_\_ $x$  trusts \_\_\_\_\_ $y$ .

$h$ : Hofthor

$i$ : Ingmar

symbolize the following sentences in FOL:

1. Hofthor trusts a vegetarian.

$$\exists x(Vx \wedge Thx)$$

2. Everyone who trusts Ingmar trusts a vegetarian.

$$\forall x[Txi \rightarrow \exists y(Txy \wedge Vy)]$$

3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.

$$\forall x[Txi \rightarrow \exists y(Txy \wedge \exists z(Tyz \wedge Vz))]$$

4. Only Ingmar knows the combination to the safe.

$$\forall x(Ki \rightarrow x = i)$$

Comment: does the English claim entail that Ingmar *does* know the combination to the safe? If so, then we should formalise this with a ' $\leftrightarrow$ '.

5. Ingmar trusts Hofthor, but no one else.

$$\forall x(Tix \leftrightarrow x = h)$$

6. The person who knows the combination to the safe is a vegetarian.

$$\exists x[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Vx]$$

7. The person who knows the combination to the safe is not a spy.

$$\exists x[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge \neg Sx]$$

Comment: the scope of negation is potentially ambiguous here; I have read it as *inner* negation.

**B.** Using the following symbolization key:

domain: cards in a standard deck

- $Bx$ : \_\_\_\_\_ $x$  is black.  
 $Cx$ : \_\_\_\_\_ $x$  is a club.  
 $Dx$ : \_\_\_\_\_ $x$  is a deuce.  
 $Jx$ : \_\_\_\_\_ $x$  is a jack.  
 $Mx$ : \_\_\_\_\_ $x$  is a man with an axe.  
 $Ox$ : \_\_\_\_\_ $x$  is one-eyed.  
 $Wx$ : \_\_\_\_\_ $x$  is wild.

symbolize each sentence in FOL:

1. All clubs are black cards.

$$\forall x(Cx \rightarrow Bx)$$

2. There are no wild cards.

$$\neg \exists x Wx$$

3. There are at least two clubs.

$$\exists x \exists y (\neg x = y \wedge Cx \wedge Cy)$$

4. There is more than one one-eyed jack.

$$\exists x \exists y (\neg x = y \wedge Jx \wedge Ox \wedge Jy \wedge Oy)$$

5. There are at most two one-eyed jacks.

$$\forall x \forall y \forall z [(Jx \wedge Ox \wedge Jy \wedge Oy \wedge Jz \wedge Oz) \rightarrow (x = y \vee x = z \vee y = z)]$$

6. There are two black jacks.

$$\exists x \exists y (\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy)$$

Comment: I am reading this as ‘there are *at least* two...’. If the suggestion was that there are *exactly* two, then a different FOL sentence would be required, namely:

$$\exists x \exists y (\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy \wedge \forall z [(Bz \wedge Jz) \rightarrow (x = z \vee y = z)])$$

7. There are four deuces.

$$\exists w \exists x \exists y \exists z (\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz)$$

Comment: I am reading this as ‘there are *at least* four...’. If the suggestion is that there are *exactly* four, then we should offer instead:

$$\exists w \exists x \exists y \exists z (\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz \wedge \forall v [Dv \rightarrow (v = w \vee v = x \vee v = y \vee v = z)])$$

8. The deuce of clubs is a black card.

$$\exists x [Dx \wedge Cx \wedge \forall y ((Dy \wedge Cy) \rightarrow x = y) \wedge Bx]$$

9. One-eyed jacks and the man with the axe are wild.

$$\forall x [(Jx \wedge Ox) \rightarrow Wx] \wedge \exists x [Mx \wedge \forall y (My \rightarrow x = y) \wedge Wx]$$

10. If the deuce of clubs is wild, then there is exactly one wild card.

$$\exists x (Dx \wedge Cx \wedge \forall y [(Dy \wedge Cy) \rightarrow x = y] \wedge Wx) \rightarrow \exists x (Wx \wedge \forall y (Wy \rightarrow x = y))$$

Comment: if there is not exactly one deuce of clubs, then the above sentence is true. Maybe that’s the wrong verdict. Perhaps the sentence should definitely be taken to imply that there is one and only one deuce of clubs, and then express a conditional about wildness. If so, then we might sym-



bolize it thus:

$$\exists x(Dx \wedge Cx \wedge \forall y[(Dy \wedge Cy) \rightarrow x = y] \wedge [Wx \rightarrow \forall y(Wy \rightarrow x = y)])$$

11. The man with the axe is not a jack.

$$\exists x[Mx \wedge \forall y(My \rightarrow x = y) \wedge \neg Jx]$$

12. The duce of clubs is not the man with the axe.

$$\exists x\exists y(Dx \wedge Cx \wedge \forall z[(Dz \wedge Cz) \rightarrow x = z] \wedge My \wedge \forall z(Mz \rightarrow y = x) \wedge \neg x = y)$$

C. Using the following symbolization key:

domain: animals in the world

$Bx$ : \_\_\_\_\_ $x$  is in Farmer Brown's field.

$Hx$ : \_\_\_\_\_ $x$  is a horse.

$Px$ : \_\_\_\_\_ $x$  is a Pegasus.

$Wx$ : \_\_\_\_\_ $x$  has wings.

symbolize the following sentences in FOL:

1. There are at least three horses in the world.

$$\exists x\exists y\exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz)$$

2. There are at least three animals in the world.

$$\exists x\exists y\exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z)$$

3. There is more than one horse in Farmer Brown's field.

$$\exists x\exists y(\neg x = y \wedge Hx \wedge Hy \wedge Bx \wedge By)$$

4. There are three horses in Farmer Brown's field.

$$\exists x\exists y\exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz \wedge Bx \wedge By \wedge Bz)$$

Comment: I have read this as 'there are *at least* three...'. If the suggestion was that there are *exactly* three, then a different FOL sentence would be required.

5. There is a single winged creature in Farmer Brown's field; any other creatures in the field must be wingless.

$$\exists x[Wx \wedge Bx \wedge \forall y((Wy \wedge By) \rightarrow x = y)]$$

6. The Pegasus is a winged horse.

$$\exists x[Px \wedge \forall y(Py \rightarrow x = y) \wedge Wx \wedge Hx]$$

7. The animal in Farmer Brown's field is not a horse.

$$\exists x[Bx \wedge \forall y(By \rightarrow x = y) \wedge \neg Hx]$$

Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.

8. The horse in Farmer Brown's field does not have wings.

$$\exists x[Hx \wedge Bx \wedge \forall y((Hy \wedge By) \rightarrow x = y) \wedge \neg Wx]$$

Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.

D. In this chapter, we symbolized 'Nick is the traitor' by ' $\exists x(Tx \wedge \forall y(Ty \rightarrow x = y) \wedge x = n)$ '. Two equally good symbolizations would be:

- $Tn \wedge \forall y(Ty \rightarrow n = y)$

This sentence requires that Nick is a traitor, and that Nick alone is a traitor. Otherwise put, there is one and only one traitor, namely, Nick. Otherwise put: Nick is the traitor.

- $\forall y(Ty \leftrightarrow y = n)$

This sentence can be understood thus: Take anything you like; now, if you chose a traitor, you chose Nick, and if you chose Nick, you chose a traitor. So there is one and only one traitor, namely, Nick, as required.

Explain why these would be equally good symbolizations.

## CHAPTER 26

# *Sentences of FOL*

A. Identify which variables are bound and which are free. We underline the bound variables, and overline the free variables.

1.  $\exists x L \underline{x} \bar{y} \wedge \forall y L y \bar{x}$
2.  $\forall x A \underline{x} \wedge B \bar{x}$
3.  $\forall x (A \underline{x} \wedge B \underline{x}) \wedge \forall y (C \bar{x} \wedge D y)$
4.  $\forall x \exists y [R x y \rightarrow (J \bar{z} \wedge K \underline{x})] \vee R y \bar{x}$
5.  $\forall x_1 (M \bar{x}_2 \leftrightarrow L \bar{x}_2 \underline{x}_1) \wedge \exists x_2 L \bar{x}_3 \underline{x}_2$

## CHAPTER 28

# *Truth in FOL*

A. Consider the following interpretation:

- The domain comprises only Corwin and Benedict
- ' $Ax$ ' is to be true of both Corwin and Benedict
- ' $Bx$ ' is to be true of Benedict only
- ' $Nx$ ' is to be true of no one
- ' $c$ ' is to refer to Corwin

Determine whether each of the following sentences is true or false in that interpretation:

- |  |       |
|--|-------|
| 1. $Bc$                                  | False |
| 2. $Ac \leftrightarrow \neg Nc$          | True  |
| 3. $Nc \rightarrow (Ac \vee Bc)$         | True  |
| 4. $\forall xAx$                         | True  |
| 5. $\forall x\neg Bx$                    | False |
| 6. $\exists x(Ax \wedge Bx)$             | True  |
| 7. $\exists x(Ax \rightarrow Nx)$        | False |
| 8. $\forall x(Nx \vee \neg Nx)$          | True  |
| 9. $\exists xBx \rightarrow \forall xAx$ | True  |

B. Consider the following interpretation:

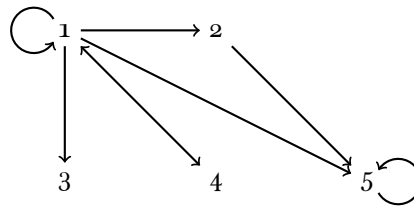
- The domain comprises only Lemmy, Courtney and Eddy
- ' $Gx$ ' is to be true of Lemmy, Courtney and Eddy.
- ' $Hx$ ' is to be true of and only of Courtney
- ' $Mx$ ' is to be true of and only of Lemmy and Eddy
- ' $c$ ' is to refer to Courtney
- ' $e$ ' is to refer to Eddy

Determine whether each of the following sentences is true or false in that interpretation:

- |         |       |
|---------|-------|
| 1. $Hc$ | True  |
| 2. $He$ | False |

- |  |       |
|--|-------|
| 3. $Mc \vee Me$                              | True  |
| 4. $Gc \vee \neg Gc$                         | True  |
| 5. $Mc \rightarrow Gc$                       | True  |
| 6. $\exists x Hx$                            | True  |
| 7. $\forall x Hx$                            | False |
| 8. $\exists x \neg Mx$                       | True  |
| 9. $\exists x (Hx \wedge Gx)$                | True  |
| 10. $\exists x (Mx \wedge Gx)$               | True  |
| 11. $\forall x (Hx \vee Mx)$                 | True  |
| 12. $\exists x Hx \wedge \exists x Mx$       | True  |
| 13. $\forall x (Hx \leftrightarrow \neg Mx)$ | True  |
| 14. $\exists x Gx \wedge \exists x \neg Gx$  | False |
| 15. $\forall x \exists y (Gx \wedge Hy)$     | True  |

C. Following the diagram conventions introduced at the end of §23, consider the following interpretation:



Determine whether each of the following sentences is true or false in that interpretation:

- |  |       |
|--|-------|
| 1. $\exists x Rxx$   | True  |
| 2. $\forall x Rxx$   | False |
| 3. $\exists x \forall y Rxy$   | True  |
| 4. $\exists x \forall y Ryx$   | False |
| 5. $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$                          | False |
| 6. $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)$                          | False |
| 7. $\exists x \forall y \neg Rxy$  | True  |
| 8. $\forall x (\exists y Rxy \rightarrow \exists y Ryx)$                                       | True  |
| 9. $\exists x \exists y (\neg x = y \wedge Rxy \wedge Ryx)$                                    | True  |
| 10. $\exists x \forall y (Rxy \leftrightarrow x = y)$  | True  |
| 11. $\exists x \forall y (Ryx \leftrightarrow x = y)$  | False |
| 12. $\exists x \exists y (\neg x = y \wedge Rxy \wedge \forall z (Rzx \leftrightarrow y = z))$ | True  |

## CHAPTER 30

# Using Interpretations

A. Show that each of the following is neither a logical truth nor a contradiction:

1.  $Da \wedge Db$
2.  $\exists xTxh$
3.  $Pm \wedge \neg\forall xPx$
4.  $\forall zJz \leftrightarrow \exists yJy$
5.  $\forall x(Wxmn \vee \exists yLxy)$
6.  $\exists x(Gx \rightarrow \forall yMy)$
7.  $\exists x(x = h \wedge x = i)$

B. Show that the following pairs of sentences are not logically equivalent.

1.  $Ja, Ka$
2.  $\exists xJx, Jm$
3.  $\forall xRxx, \exists xRxx$
4.  $\exists xPx \rightarrow Qc, \exists x(Px \rightarrow Qc)$
5.  $\forall x(Px \rightarrow \neg Qx), \exists x(Px \wedge \neg Qx)$
6.  $\exists x(Px \wedge Qx), \exists x(Px \rightarrow Qx)$
7.  $\forall x(Px \rightarrow Qx), \forall x(Px \wedge Qx)$
8.  $\forall x\exists yRxy, \exists x\forall yRxy$
9.  $\forall x\exists yRxy, \forall x\exists yRyx$

C. Show that the following sentences are jointly consistent:

1.  $Ma, \neg Na, Pa, \neg Qa$
2.  $Lee, Leg, \neg Lge, \neg Lgg$
3.  $\neg(Ma \wedge \exists xAx), Ma \vee Fa, \forall x(Fx \rightarrow Ax)$
4.  $Ma \vee Mb, Ma \rightarrow \forall x\neg Mx$
5.  $\forall yGy, \forall x(Gx \rightarrow Hx), \exists y\neg Iy$
6.  $\exists x(Bx \vee Ax), \forall x\neg Cx, \forall x[(Ax \wedge Bx) \rightarrow Cx]$

7.  $\exists xXx, \exists xYx, \forall x(Xx \leftrightarrow \neg Yx)$
8.  $\forall x(Px \vee Qx), \exists x\neg(Qx \wedge Px)$
9.  $\exists z(Nz \wedge Ozz), \forall x\forall y(Oxy \rightarrow Oyx)$
10.  $\neg\exists x\forall yRxy, \forall x\exists yRxy$
11.  $\neg Raa, \forall x(x = a \vee Rxa)$
12.  $\forall x\forall y\forall z[(x = y \vee y = z) \vee x = z], \exists x\exists y \neg x = y$
13.  $\exists x\exists y((Zx \wedge Zy) \wedge x = y), \neg Zd, d = e$

**D.** Show that the following arguments are invalid:

1.  $\forall x(Ax \rightarrow Bx) \therefore \exists xBx$
2.  $\forall x(Rx \rightarrow Dx), \forall x(Rx \rightarrow Fx) \therefore \exists x(Dx \wedge Fx)$
3.  $\exists x(Px \rightarrow Qx) \therefore \exists xPx$
4.  $Na \wedge Nb \wedge Nc \therefore \forall xNx$
5.  $Rde, \exists xRxd \therefore Red$
6.  $\exists x(Ex \wedge Fx), \exists xFx \rightarrow \exists xGx \therefore \exists x(Ex \wedge Gx)$
7.  $\forall xOxc, \forall xOcx \therefore \forall xOxx$
8.  $\exists x(Jx \wedge Kx), \exists x\neg Kx, \exists x\neg Jx \therefore \exists x(\neg Jx \wedge \neg Kx)$
9.  $Lab \rightarrow \forall xLxb, \exists xLxb \therefore Lbb$
10.  $\forall x(Dx \rightarrow \exists yTyx) \therefore \exists y\exists z \neg y = z$

CHAPTER 32

# Basic rules for FOL

A. The following two ‘proofs’ are *incorrect*. Explain why both are incorrect. Also, provide interpretations which would invalidate the fallacious argument forms the ‘proofs’ enshrine:

1	$\forall xRxx$	
2	$Raa$	$\forall E$ 1
3	$\forall yRay$	$\forall I$ 2
4	$\forall x\forall yRxy$	$\forall I$ 3

1	$\forall x\exists yRxy$	
2	$\exists yRay$	$\forall E$ 1
3	$Raa$	
4	$\exists xRxx$	$\exists I$ 3
5	$\exists xRxx$	$\exists E$ 2, 3–4

When using  $\forall I$ , you must replace *all* names with the new variable. So line 3 is bogus. As a counterinterpretation, consider the following:



The instantiating constant, ‘a’, occurs in the line (line 2) to which  $\exists E$  is to be applied on line 5. So the use of  $\exists E$  on line 5 is bogus. As a counterinterpretation, consider the following:



B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs.

1	$\forall x\exists y(Rxy \vee Ryx)$	
2	$\forall x\neg Rmx$	
3	$\exists y(Rmy \vee Rym)$	$\forall E$ 1
4	$Rma \vee Ram$	
5	$\neg Rma$	$\forall E$ 2
6	$Ram$	$DS$ 4, 5     77
7	$\exists xRxm$	$\exists I$ 6
8	$\exists xRxm$	$\exists E$ 3, 4–7



<table style="border-collapse: collapse; width: 100%;"> <tr><td style="border-right: 1px solid black; padding-right: 5px;">1</td><td style="padding-right: 5px;"><math>\forall x(\exists yLxy \rightarrow \forall zLzx)</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">2</td><td style="padding-right: 5px;"><math>Lab</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">3</td><td style="padding-right: 5px;"><math>\exists yLay \rightarrow \forall zLza</math></td><td style="padding-left: 5px;"><math>\forall E</math> 1</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">4</td><td style="padding-right: 5px;"><math>\exists yLay</math></td><td style="padding-left: 5px;"><math>\exists I</math> 2</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">5</td><td style="padding-right: 5px;"><math>\forall zLza</math></td><td style="padding-left: 5px;"><math>\rightarrow E</math> 3, 4</td></tr> <tr><td style="border-right: 1px solid black; 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C. In §22 problem A, we considered fifteen syllogistic figures of Aristotelian logic. Provide proofs for each of the argument forms. NB: You will find it *much* easier if you symbolize (for example) ‘No F is G’ as ‘ $\forall x(Fx \rightarrow \neg Gx)$ ’.

I shall prove the four Figure I syllogisms; the rest are *extremely* similar.

#### Barbara

1	$\forall x(Gx \rightarrow Fx)$	
2	$\forall x(Hx \rightarrow Gx)$	
3	$Ga \rightarrow Fa$	$\forall E$ 1
4	$Ha \rightarrow Ga$	$\forall E$ 2
5	$Ha$	
6	$Ga$	$\rightarrow E$ 4, 5
7	$Fa$	$\rightarrow E$ 3, 6
8	$Ha \rightarrow Fa$	$\rightarrow I$ 5–7
9	$\forall x(Hx \rightarrow Fx)$	$\forall I$ 8

**Celerant** is exactly as Barbara, replacing ‘ $F$ ’ with ‘ $\neg F$ ’ throughout.

#### Ferio

1	$\forall x(Gx \rightarrow \neg Fx)$	
2	$\exists x(Hx \wedge Gx)$	
3	$Ha \wedge Ga$	
4	$Ha$	$\wedge E$ 3
5	$Ga$	$\wedge E$ 3
6	$Ga \rightarrow \neg Fa$	$\forall E$ 1
7	$\neg Fa$	$\rightarrow E$ 6, 5
8	$Ha \wedge \neg Fa$	$\wedge I$ 4, 7
9	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 2, 3–9

**Darii** is exactly as Ferio, replacing ‘ $\neg F$ ’ with ‘ $F$ ’ throughout.

D. Aristotle and his successors identified other syllogistic forms which depended upon ‘existential import’. Symbolize each of the following argument forms in FOL and offer proofs.

- **Barbari.** Something is H. All G are F. All H are G. So: Some H is F  
 $\exists xHx, \forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge Fx)$

1	$\exists xHx$	
2	$\forall x(Gx \rightarrow Fx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$H a$	
5	$H a \rightarrow G a$	$\forall E 3$
6	$G a$	$\rightarrow E 5, 4$
7	$G a \rightarrow F a$	$\forall E 2$
8	$F a$	$\rightarrow E 7, 6$
9	$H a \wedge F a$	$\wedge I 4, 8$
10	$\exists x(Hx \wedge Fx)$	$\exists I 9$
11	$\exists x(Hx \wedge Fx)$	$\exists E 1, 4-10$

- **Celarent.** Something is H. No G are F. All H are G. So: Some H is not F  
 $\exists xHx, \forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

Proof is exactly as for Barbari, replacing 'F' with ' $\neg F$ ' throughout.

- **Cesaro.** Something is H. No F are G. All H are G. So: Some H is not F.  
 $\exists xHx, \forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists xHx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$H a$	
5	$H a \rightarrow G a$	$\forall E 3$
6	$G a$	$\rightarrow E 5, 4$
7	$F a \rightarrow \neg G a$	$\forall E 2$
8	$F a$	
9	$\neg G a$	$\rightarrow E 7, 8$
10	$\perp$	$\neg E 6, 9$
11	$\neg F a$	$\neg I 8-10$
12	$H a \wedge \neg F a$	$\wedge I 4, 11$
13	$\exists x(Hx \wedge \neg Fx)$	$\exists I 12$
14	$\exists x(Hx \wedge \neg Fx)$	$\exists E 1, 4-13$

- **Camestros.** Something is H. All F are G. No H are G. So: Some H is not F.

$$\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Hx \rightarrow \neg Gx)$	
4	$H a$	
5	$H a \rightarrow \neg G a$	$\forall E$ 3
6	$\neg G a$	$\rightarrow E$ 5, 4
7	$F a \rightarrow G a$	$\forall E$ 2
8	$\neg F a$	MT 7, 6
9	$H a \wedge \neg F a$	$\wedge I$ 4, 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 1, 4–10

- **Felapton.** Something is G. No G are F. All G are H. So: Some H is not F.

$$\exists x Gx, \forall x(Gx \rightarrow \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists x Gx$	
2	$\forall x(Gx \rightarrow \neg Fx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$G a$	
5	$G a \rightarrow H a$	$\forall E$ 3
6	$H a$	$\rightarrow E$ 5, 4
7	$G a \rightarrow \neg F a$	$\forall E$ 2
8	$\neg F a$	$\rightarrow E$ 7, 4
9	$H a \wedge \neg F a$	$\wedge I$ 6, 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 1, 4–10

- **Darapti.** Something is G. All G are F. All G are H. So: Some H is F.

$$\exists x Gx, \forall x(Gx \rightarrow Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$$

Proof is exactly as for Felapton, replacing ' $\neg F$ ' with ' $F$ ' throughout.

- **Calemos.** Something is H. All F are G. No G are H. So: Some H is not F.

$$\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Gx \rightarrow \neg Hx)$	
4	$H a$	
5	$G a \rightarrow \neg H a$	$\forall E$ 3
6	$G a$	
7	$\neg H a$	$\rightarrow E$ 5, 6
8	$\perp$	$\neg E$ 4, 7
9	$\neg G a$	$\neg I$ 6–8
10	$F a \rightarrow G a$	$\forall E$ 2
11	$\neg F a$	MT 10, 9
12	$H a \wedge \neg F a$	$\wedge I$ 4, 11
13	$\exists x(Hx \wedge Fx)$	$\exists I$ 12
14	$\exists x(Hx \wedge Fx)$	$\exists E$ 1, 4–13

- **Fesapo.** Something is G. No F is G. All G are H. So: Some H is not F.

$$\exists x Gx, \forall x(Fx \rightarrow \neg Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists x Gx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$G a$	
5	$G a \rightarrow H a$	$\forall E$ 3
6	$H a$	$\rightarrow E$ 5, 4
7	$F a \rightarrow \neg G a$	$\forall E$ 2
8	$F a$	
9	$\neg G a$	$\rightarrow E$ 7, 8
10	$\perp$	$\neg E$ 4, 9
11	$\neg F a$	$\neg I$ 8–10
12	$H a \wedge \neg F a$	$\wedge I$ 6, 11
13	$\exists x(Hx \wedge Fx)$	$\exists I$ 12
14	$\exists x(Hx \wedge Fx)$	$\exists E$ 1, 4–13

- **Bamalip.** Something is F. All F are G. All G are H. So: Some H are F.

$$\exists xFx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$$

1	$\exists xFx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$Fa$	
5	$Fa \rightarrow Ga$	$\forall E$ 2
6	$Ga$	$\rightarrow E$ 5, 4
7	$Ga \rightarrow Ha$	$\forall E$ 3
8	$Ha$	$\rightarrow E$ 7, 6
9	$Ha \wedge Fa$	$\wedge I$ 8, 4
10	$\exists x(Hx \wedge Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge Fx)$	$\exists E$ 1, 4–10

**E.** Provide a proof of each claim.

1.  $\vdash \forall xFx \vee \neg \forall xFx$

1	$\forall xFx$	
2	$\forall xFx \vee \neg \forall xFx$	$\vee I$ 1
3	$\neg \forall xFx$	
4	$\forall xFx \vee \neg \forall xFx$	$\vee I$ 3
5	$\forall xFx \vee \neg \forall xFx$	LEM 1–2, 3–4

2.  $\vdash \forall z(Pz \vee \neg Pz)$

1	$Pa$	
2	$Pa \vee \neg Pa$	$\vee I$ 1
3	$\neg Pa$	
4	$Pa \vee \neg Pa$	$\vee I$ 3
5	$Pa \vee \neg Pa$	LEM 1–2, 3–4
6	$\forall x(Px \vee \neg Px)$	$\forall I$ 5

3.  $\forall x(Ax \rightarrow Bx), \exists xAx \vdash \exists xBx$

1		$\forall x(Ax \rightarrow Bx)$	
2		$\exists xAx$	
<hr/>			
3			$Aa$
4			$Aa \rightarrow Ba$ $\forall E$ 1
5			$Ba$ $\rightarrow E$ 4, 3
6			$\exists xBx$ $\exists I$ 5
7		$\exists xBx$	$\exists E$ 2, 3-6

4.  $\forall x(Mx \leftrightarrow Nx), Ma \wedge \exists xRxa \vdash \exists xNx$

1	$\forall x(Mx \leftrightarrow Nx)$	
2	$Ma \wedge \exists xRxa$	
3	$Ma$	$\wedge E$ 2
4	$Ma \leftrightarrow Na$	$\forall E$ 1
5	$Na$	$\leftrightarrow E$ 4, 3
6	$\exists xNx$	$\exists I$ 5

5.  $\forall x\forall yGxy \vdash \exists xGxx$

1	$\forall x\forall yGxy$	
2	$\forall yGay$	$\forall E$ 1
3	$Gaa$	$\forall E$ 2
4	$\exists xGxx$	$\exists I$ 3

6.  $\vdash \forall xRxx \rightarrow \exists x\exists yRxy$

1	$\forall xRxx$	
2	$Raa$	$\forall E$ 1
3	$\exists yRay$	$\exists I$ 2
4	$\exists x\exists yRxy$	$\exists I$ 3
5	$\forall xRxx \rightarrow \exists x\exists yRxy$	$\rightarrow I$ 1-4

7.  $\vdash \forall y\exists x(Qy \rightarrow Qx)$

1	$Qa$	
2	$Qa$	R 1
3	$Qa \rightarrow Qa$	$\rightarrow I$ 1-2
4	$\exists x(Qa \rightarrow Qx)$	$\exists I$ 3
5	$\forall y\exists x(Qy \rightarrow Qx)$	$\forall I$ 4

8.  $Na \rightarrow \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb \vdash \neg Na$

1		$Na \rightarrow \forall x(Mx \leftrightarrow Ma)$	
2		$Ma$	
3		$\neg Mb$	
<hr/>			
4		$Na$	
5		$\forall x(Mx \leftrightarrow Ma)$	$\rightarrow E$ 1, 4
6		$Mb \leftrightarrow Ma$	$\forall E$ 5
7		$Mb$	$\leftrightarrow E$ 6, 2
8		$\perp$	$\neg E$ 7, 3
9		$\neg Na$	$\neg I$ 4–8



9.  $\forall x\forall y(Gxy \rightarrow Gyx) \vdash \forall x\forall y(Gxy \leftrightarrow Gyx)$

1	$\forall x\forall y(Gxy \rightarrow Gyx)$	
2	$Gab$	
3	$\forall y(Gay \rightarrow Gya)$	$\forall E$ 1
4	$Gab \rightarrow Gba$	$\forall E$ 3
5	$Gba$	$\rightarrow E$ 4, 2
6	$Gba$	
7	$\forall y(Gby \rightarrow Gyb)$	$\forall E$ 1
8	$Gba \rightarrow Gab$	$\forall E$ 7
9	$Gab$	$\rightarrow E$ 8, 6
10	$Gab \leftrightarrow Gba$	$\leftrightarrow I$ 2–5, 6–9
11	$\forall y(Gay \leftrightarrow Gya)$	$\forall I$ 10
12	$\forall x\forall y(Gxy \leftrightarrow Gyx)$	$\forall I$ 11

10.  $\forall x(\neg Mx \vee Ljx), \forall x(Bx \rightarrow Ljx), \forall x(Mx \vee Bx) \vdash \forall xLjx$

1	$\forall x(\neg Mx \vee Ljx)$	
2	$\forall x(Bx \rightarrow Ljx)$	
3	$\forall x(Mx \vee Bx)$	
4	$\neg Ma \vee Lja$	$\forall E$ 1
5	$Ba \rightarrow Lja$	$\forall E$ 2
6	$Ma \vee Ba$	$\forall E$ 3
7	$\neg Ma$	
8	$Ba$	$DS$ 6, 7
9	$Lja$	$\rightarrow E$ 5, 8
10	$Lja$	
11	$Lja$	$R$ 10
12	$Lja$	$\forall E$ 4, 7–9, 10–11
13	$\forall xLjx$	$\forall I$ 12

F. Write a symbolization key for the following argument, symbolize it, and prove it:

There is someone who likes everyone who likes everyone that she likes. Therefore, there is someone who likes herself.

Symbolization key:

domain: all people

$Lxy$ : \_\_\_\_\_ $x$  likes \_\_\_\_\_ $y$

$\exists x\forall y(\forall z(Lxz \rightarrow Lyz) \rightarrow Lxy) \therefore \exists xLxx$

1	$\exists x\forall y(\forall z(Lxz \rightarrow Lyz) \rightarrow Lxy)$																			
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall y(\forall z(Laz \rightarrow Lyz) \rightarrow Lay)</math> </td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Lac</math> </td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Lac</math> </td> <td style="padding-left: 10px;">R 4</td> </tr> </table> </td> <td style="padding-left: 10px;"><math>Lac \rightarrow Lac</math></td> <td style="padding-left: 10px;"><math>\rightarrow I</math> 4–5</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall z(Laz \rightarrow Laz) \rightarrow Laa</math> </td> <td style="padding-left: 10px;"><math>\forall I</math> 6</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Laa</math> </td> <td style="padding-left: 10px;"><math>\rightarrow E</math> 3, 7</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\exists xLxx</math> </td> <td style="padding-left: 10px;"><math>\exists I</math> 8</td> <td></td> </tr> </table>	$\forall y(\forall z(Laz \rightarrow Lyz) \rightarrow Lay)$		<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Lac</math> </td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Lac</math> </td> <td style="padding-left: 10px;">R 4</td> </tr> </table>	$Lac$		$Lac$	R 4	$Lac \rightarrow Lac$	$\rightarrow I$ 4–5	$\forall z(Laz \rightarrow Laz) \rightarrow Laa$	$\forall I$ 6		$Laa$	$\rightarrow E$ 3, 7		$\exists xLxx$	$\exists I$ 8		$\forall E$ 2
$\forall y(\forall z(Laz \rightarrow Lyz) \rightarrow Lay)$																				
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$Laa$	$\rightarrow E$ 3, 7																			
$\exists xLxx$	$\exists I$ 8																			
10	$\exists xLxx$	$\exists E$ 1, 2–9																		

**G.** Show that each pair of sentences is provably equivalent.

1.  $\forall x(Ax \rightarrow \neg Bx), \neg \exists x(Ax \wedge Bx)$
2.  $\forall x(\neg Ax \rightarrow Bx), \forall xAx \vee Bx$
3.  $\exists xPx \rightarrow Qc, \forall x(Px \rightarrow Qc)$

**H.** For each of the following pairs of sentences: If they are provably equivalent, give proofs to show this. If they are not, construct an interpretation to show that they are not logically equivalent.

1.  $\forall xPx \rightarrow Qc, \forall x(Px \rightarrow Qc)$  Not logically equivalent  
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ $c$ ’ name 1. Let ‘ $Px$ ’ be true of and only of 1. Let ‘ $Qx$ ’ be true of, and only of, 2.
2.  $\forall x\forall y\forall zBxyz, \forall xBxxx$  Not logically equivalent  
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ $Bxyz$ ’ be true of, and only of,  $\langle 1,1,1 \rangle$  and  $\langle 2,2,2 \rangle$ .
3.  $\forall x\forall yDxy, \forall y\forall xDxy$  Provably equivalent

1	$\forall x\forall yDxy$									
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall yDay</math> </td> <td style="padding-left: 10px;"><math>\forall E</math> 1</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Dab</math> </td> <td style="padding-left: 10px;"><math>\forall E</math> 2</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall xDxb</math> </td> <td style="padding-left: 10px;"><math>\forall I</math> 3</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall y\forall xDxy</math> </td> <td style="padding-left: 10px;"><math>\forall I</math> 4</td> </tr> </table>	$\forall yDay$	$\forall E$ 1	$Dab$	$\forall E$ 2	$\forall xDxb$	$\forall I$ 3	$\forall y\forall xDxy$	$\forall I$ 4	
$\forall yDay$	$\forall E$ 1									
$Dab$	$\forall E$ 2									
$\forall xDxb$	$\forall I$ 3									
$\forall y\forall xDxy$	$\forall I$ 4									

1	$\forall y\forall xDxy$									
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall xDxa</math> </td> <td style="padding-left: 10px;"><math>\forall E</math> 1</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>Dba</math> </td> <td style="padding-left: 10px;"><math>\forall E</math> 2</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall yDby</math> </td> <td style="padding-left: 10px;"><math>\forall I</math> 3</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <math>\forall x\forall yDxy</math> </td> <td style="padding-left: 10px;"><math>\forall I</math> 4</td> </tr> </table>	$\forall xDxa$	$\forall E$ 1	$Dba$	$\forall E$ 2	$\forall yDby$	$\forall I$ 3	$\forall x\forall yDxy$	$\forall I$ 4	
$\forall xDxa$	$\forall E$ 1									
$Dba$	$\forall E$ 2									
$\forall yDby$	$\forall I$ 3									
$\forall x\forall yDxy$	$\forall I$ 4									

4.  $\exists x\forall yDxy, \forall y\exists xDxy$  Not logically equivalent  
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ $Dxy$ ’ hold of and only of  $\langle 1,2 \rangle$  and  $\langle 2,1 \rangle$ . This is depicted thus:



5.  $\forall x(Rca \leftrightarrow Rxa), Rca \leftrightarrow \forall xRxa$  Not logically equivalent  
 Counter-interpretation, consider the following diagram, allowing ‘ $a$ ’ to name 1 and ‘ $c$ ’ to name 2:



I. For each of the following arguments: If it is valid in FOL, give a proof. If it is invalid, construct an interpretation to show that it is invalid.

1.  $\exists y\forall xRxy \therefore \forall x\exists yRxy$  Valid

1	$\exists y\forall xRxy$	
2	$\forall xRxa$	
3	$Rba$	$\forall E$ 2
4	$\exists yRby$	$\exists I$ 3
5	$\exists yRby$	$\exists E$ 1, 2–4
6	$\forall x\exists yRxy$	$\forall I$ 5

2.  $\forall x\exists yRxy \therefore \exists y\forall xRxy$  Not valid  
 Counter interpretation: let the domain be the numbers 1 and 2. Let ‘ $Rxy$ ’ be true of 1 and 2, and of 2 and 1 (but not 1 and itself or 2 and itself).

3.  $\exists x(Px \wedge \neg Qx) \therefore \forall x(Px \rightarrow \neg Qx)$  Not valid  
 Counter interpretation: let the domain be the numbers 1 and 2. Let ‘ $Px$ ’ be true of everything in the domain. Let ‘ $Qx$ ’ be true of, and only of, 2.

4.  $\forall x(Sx \rightarrow Ta), Sd \therefore Ta$  Valid

1	$\forall x(Sx \rightarrow Ta)$	
2	$Sd$	
3	$Sd \rightarrow Ta$	$\forall E$ 1
4	$Ta$	$\rightarrow E$ 3, 2

5.  $\forall x(Ax \rightarrow Bx), \forall x(Bx \rightarrow Cx) \therefore \forall x(Ax \rightarrow Cx)$  Valid

- |   |                                |                      |
|---|--------------------------------|----------------------|
| 1 | $\forall x(Ax \rightarrow Bx)$ |                      |
| 2 | $\forall x(Bx \rightarrow Cx)$ |                      |
|   |                                |                      |
| 3 | $Aa \rightarrow Ba$            | $\forall E$ 1        |
| 4 | $Ba \rightarrow Ca$            | $\forall E$ 2        |
| 5 | $Aa$                           |                      |
| 6 | $Ba$                           | $\rightarrow E$ 3, 5 |
| 7 | $Ca$                           | $\rightarrow E$ 4, 6 |
| 8 | $Aa \rightarrow Ca$            | $\rightarrow I$ 5-7  |
| 9 | $\forall x(Ax \rightarrow Cx)$ | $\forall I$ 8        |
6.  $\exists x(Dx \vee Ex), \forall x(Dx \rightarrow Fx) \therefore \exists x(Dx \wedge Fx)$  Invalid  
 Counter-interpretation: let the domain be the number 1. Let ‘ $Dx$ ’ hold of nothing. Let both ‘ $Ex$ ’ and ‘ $Fx$ ’ hold of everything.
7.  $\forall x\forall y(Rxy \vee Ryx) \therefore Rjj$  Valid
- |   |                                    |                         |
|---|------------------------------------|-------------------------|
| 1 | $\forall x\forall y(Rxy \vee Ryx)$ |                         |
|   |                                    |                         |
| 2 | $\forall y(Rjy \vee Ryj)$          | $\forall E$ 1           |
| 3 | $Rjj \vee Rjj$                     | $\forall E$ 2           |
| 4 | $Rjj$                              |                         |
| 5 | $Rjj$                              | R 4                     |
| 6 | $Rjj$                              |                         |
| 7 | $Rjj$                              | R 6                     |
| 8 | $Rjj$                              | $\forall E$ 3, 4-5, 6-7 |
8.  $\exists x\exists y(Rxy \vee Ryx) \therefore Rjj$  Invalid  
 Counter-interpretation: consider the following diagram, allowing ‘ $j$ ’ to name 2.
- 
9.  $\forall xPx \rightarrow \forall xQx, \exists x\neg Px \therefore \exists x\neg Qx$  Invalid  
 Counter-interpretation: let the domain be the number 1. Let ‘ $Px$ ’ be true of nothing. Let ‘ $Qx$ ’ be true of everything.
10.  $\exists xMx \rightarrow \exists xNx, \neg\exists xNx \therefore \forall x\neg Mx$  Valid

1		$\exists xMx \rightarrow \exists xNx$	
2		$\neg\exists xNx$	
<hr/>			
3		$Ma$	
<hr/>			
4		$\exists xMx$	$\exists I$ 3
5		$\exists xNx$	$\rightarrow E$ 1, 4
6		$\perp$	$\neg E$ 5, 2
7		$\neg Ma$	$\neg I$ 3-6
8		$\forall x\neg Mx$	$\forall I$ 7

CHAPTER 33

# Conversion of quantifiers

A. Show in each case that the sentences are provably inconsistent:

1.  $Sa \rightarrow Tm, Tm \rightarrow Sa, Tm \wedge \neg Sa$

1	$Sa \rightarrow Tm$	
2	$Tm \rightarrow Sa$	
3	$Tm \wedge \neg Sa$	
4	$Tm$	$\wedge E$ 3
5	$\neg Sa$	$\wedge E$ 3
6	$Sa$	$\rightarrow E$ 2, 4
7	$\perp$	$\neg E$ 5, 6

2.  $\neg \exists x Rxa, \forall x \forall y Ryx$

1	$\neg \exists x Rxa$	
2	$\forall x \forall y Ryx$	
3	$\forall x \neg Rxa$	CQ 1
4	$\neg Rba$	$\forall E$ 3
5	$\forall y Rya$	$\forall E$ 2
6	$Rba$	$\forall E$ 5
7	$\perp$	$\neg E$ 6, 4

3.  $\neg \exists x \exists y Lxy, Laa$

1	$\neg\exists x\exists yLxy$	
2	$Laa$	
3	$\forall x\neg\exists yLxy$	CQ 1
4	$\neg\exists yLay$	$\forall E$ 3
5	$\forall y\neg Lay$	CQ 4
6	$\neg Laa$	$\forall E$ 5
7	$\perp$	$\neg E$ 2, 6

4.  $\forall x(Px \rightarrow Qx), \forall z(Pz \rightarrow Rz), \forall yPy, \neg Qa \wedge \neg Rb$

1	$\forall x(Px \rightarrow Qx)$	
2	$\forall z(Pz \rightarrow Rz)$	
3	$\forall yPy$	
4	$\neg Qa \wedge \neg Rb$	
5	$\neg Qa$	$\wedge E$ 4
6	$Pa \rightarrow Qa$	$\forall E$ 1
7	$\neg Pa$	MT 6, 5
8	$Pa$	$\forall E$ 3
9	$\perp$	$\neg E$ 8, 7

**B.** Show that each pair of sentences is provably equivalent:

1.  $\forall x(Ax \rightarrow \neg Bx), \neg\exists x(Ax \wedge Bx)$

1	$\forall x(Ax \rightarrow \neg Bx)$		1	$\neg\exists x(Ax \wedge Bx)$	
2	$\exists x(Ax \wedge Bx)$		2	$\forall x\neg(Ax \wedge Bx)$	CQ 1
3	$Aa \wedge Ba$		3	$\neg(Aa \wedge Ba)$	$\forall E$ 2
4	$Aa$	$\wedge E$ 3	4	$Aa$	
5	$Ba$	$\wedge E$ 3	5	$Ba$	
6	$Aa \rightarrow \neg Ba$	$\forall E$ 1	6	$Aa \wedge Ba$	$\wedge I$ 4, 5
7	$\neg Ba$	$\rightarrow E$ 6, 4	7	$\perp$	$\neg E$ 6, 3
8	$\perp$	$\neg E$ 5, 7	8	$\neg Ba$	$\neg I$ 5-7
9	$\perp$	$\exists E$ 2, 3-8	9	$Aa \rightarrow \neg Ba$	$\rightarrow I$ 4-8
10	$\neg\exists x(Ax \wedge Bx)$	$\neg I$ 2-9	10	$\forall x(Ax \rightarrow \neg Bx)$	$\forall I$ 9

2.  $\forall x(\neg Ax \rightarrow Bd), \forall xAx \vee Bd$

<table style="width: 100%; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding-right: 5px;">1</td><td style="border-bottom: 1px solid black; padding: 2px 5px;"><math>\forall x(\neg Ax \rightarrow Bd)</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">2</td><td style="padding: 2px 5px;"><math>\neg Aa \rightarrow Bd</math></td><td style="padding-left: 5px;">VE 1</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">3</td><td style="border-left: 1px solid black; border-bottom: 1px solid black; padding: 2px 5px;"><math>Bd</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">4</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>\forall xAx \vee Bd</math></td><td style="padding-left: 5px;">VI 6</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">5</td><td style="border-left: 1px solid black; border-bottom: 1px solid black; padding: 2px 5px;"><math>\neg Bd</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">6</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>\neg\neg Aa</math></td><td style="padding-left: 5px;">MT 2, 5</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">7</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>Aa</math></td><td style="padding-left: 5px;">DNE 6</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">8</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>\forall xAx</math></td><td style="padding-left: 5px;">VE 7</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">9</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>\forall xAx \vee Bd</math></td><td style="padding-left: 5px;">VI 8</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">10</td><td style="padding: 2px 5px;"><math>\forall xAx \vee Bd</math></td><td style="padding-left: 5px;">LEM 3–4, 5–9</td></tr> </table>	1	$\forall x(\neg Ax \rightarrow Bd)$		2	$\neg Aa \rightarrow Bd$	VE 1	3	$Bd$		4	$\forall xAx \vee Bd$	VI 6	5	$\neg Bd$		6	$\neg\neg Aa$	MT 2, 5	7	$Aa$	DNE 6	8	$\forall xAx$	VE 7	9	$\forall xAx \vee Bd$	VI 8	10	$\forall xAx \vee Bd$	LEM 3–4, 5–9	<table style="width: 100%; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding-right: 5px;">1</td><td style="border-bottom: 1px solid black; padding: 2px 5px;"><math>\forall xAx \vee Bd</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">2</td><td style="border-left: 1px solid black; border-bottom: 1px solid black; padding: 2px 5px;"><math>\neg Aa</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">3</td><td style="border-left: 1px solid black; border-left: 1px solid black; border-bottom: 1px solid black; padding: 2px 5px;"><math>\forall xAx</math></td><td></td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">4</td><td style="border-left: 1px solid black; border-left: 1px solid black; padding: 2px 5px;"><math>Aa</math></td><td style="padding-left: 5px;">VE 3</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">5</td><td style="border-left: 1px solid black; border-left: 1px solid black; padding: 2px 5px;"><math>\perp</math></td><td style="padding-left: 5px;"><math>\neg</math>E 4, 2</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">6</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>\neg\forall xAx</math></td><td style="padding-left: 5px;"><math>\neg</math>I 3–5</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">7</td><td style="border-left: 1px solid black; padding: 2px 5px;"><math>Bd</math></td><td style="padding-left: 5px;">DS 1, 6</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">8</td><td style="padding: 2px 5px;"><math>\neg Aa \rightarrow Bd</math></td><td style="padding-left: 5px;"><math>\rightarrow</math>I 2–7</td></tr> <tr><td style="border-right: 1px solid black; padding-right: 5px;">9</td><td style="padding: 2px 5px;"><math>\forall x(Ax \rightarrow Bd)</math></td><td style="padding-left: 5px;">VI 8</td></tr> </table>	1	$\forall xAx \vee Bd$		2	$\neg Aa$		3	$\forall xAx$		4	$Aa$	VE 3	5	$\perp$	$\neg$ E 4, 2	6	$\neg\forall xAx$	$\neg$ I 3–5	7	$Bd$	DS 1, 6	8	$\neg Aa \rightarrow Bd$	$\rightarrow$ I 2–7	9	$\forall x(Ax \rightarrow Bd)$	VI 8
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**C.** In §22, we considered what happens when we move quantifiers ‘across’ various logical operators. Show that each pair of sentences is provably equivalent:

1.  $\forall x(Fx \wedge Ga), \forall xFx \wedge Ga$

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2.  $\exists x(Fx \vee Ga), \exists xFx \vee Ga$

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3.  $\forall x(Ga \rightarrow Fx), Ga \rightarrow \forall xFx$ 

1	$\forall x(Ga \rightarrow Fx)$	
2	$Ga \rightarrow Fb$	$\forall E$ 1
3	$Ga$	
4	$Fb$	$\rightarrow E$ 2, 3
5	$\forall xFx$	$\forall I$ 4
6	$Ga \rightarrow \forall xFx$	$\rightarrow I$ 3–5

1	$Ga \rightarrow \forall xFx$	
2	$Ga$	
3	$\forall xFx$	$\rightarrow E$ 1, 2
4	$Fb$	$\forall E$ 3
5	$Ga \rightarrow Fb$	$\rightarrow I$ 2–4
6	$\forall x(Ga \rightarrow Fx)$	$\forall I$ 5

4.  $\forall x(Fx \rightarrow Ga), \exists xFx \rightarrow Ga$ 

1	$\forall x(Fx \rightarrow Ga)$	
2	$\exists xFx$	
3	$Fb$	
4	$Fb \rightarrow Ga$	$\forall E$ 1
5	$Ga$	$\rightarrow E$ 4, 3
6	$Ga$	$\exists E$ 2, 3–5
7	$\exists xFx \rightarrow Ga$	$\rightarrow I$ 2–6

1	$\exists xFx \rightarrow Ga$	
2	$Fb$	
3	$\exists xFx$	$\exists I$ 2
4	$Ga$	$\rightarrow E$ 1, 3
5	$Fb \rightarrow Ga$	$\rightarrow I$ 2–4
6	$\forall x(Fx \rightarrow Ga)$	$\forall I$ 5

5.  $\exists x(Ga \rightarrow Fx), Ga \rightarrow \exists xFx$ 

1	$\exists x(Ga \rightarrow Fx)$	
2	$Ga$	
3	$Ga \rightarrow Fb$	
4	$Fb$	$\rightarrow E$ 3, 2
5	$\exists xFx$	$\exists I$ 4
6	$\exists xFx$	$\exists E$ 1, 3–5
7	$Ga \rightarrow \exists xFx$	$\rightarrow I$ 2–6

1	$Ga \rightarrow \exists xFx$	
2	$Ga$	
3	$\exists xFx$	
4	$Fb$	
5	$Ga$	
6	$Fb$	R 4
7	$Ga \rightarrow Fb$	$\rightarrow I$ 5–6
8	$\exists x(Ga \rightarrow Fx)$	$\exists I$ 7
9	$\exists x(Ga \rightarrow Fx)$	$\exists E$ 3, 4–8
10	$\neg Ga$	
11	$Ga$	
12	$\perp$	$\neg E$ 11, 10
13	$Fb$	X 12
14	$Ga \rightarrow Fb$	$\rightarrow E$ 11–13
15	$\exists x(Ga \rightarrow Fx)$	$\exists I$ 14
16	$\exists x(Ga \rightarrow Fx)$	LEM 2–9, 10–15

6.  $\exists x(Fx \rightarrow Ga), \forall xFx \rightarrow Ga$ 

1	$\exists x(Fx \rightarrow Ga)$	
2	$\forall xFx$	
3	$Fb \rightarrow Ga$	
4	$Fb$	$\forall E$ 2
5	$Ga$	$\rightarrow E$ 3, 4
6	$Ga$	$\exists E$ 1, 3–5
7	$\forall xFx \rightarrow Ga$	$\rightarrow I$ 2–6

1	$\forall xFx \rightarrow Ga$	
2	$\forall xFx$	
3	$Ga$	$\rightarrow E$ 1, 2
4	$Fb$	
5	$Ga$	R 3
6	$Fb \rightarrow Ga$	$\rightarrow I$ 4–5
7	$\exists x(Fx \rightarrow Ga)$	$\exists I$ 6
8	$\neg \forall xFx$	
9	$\exists x \neg Fx$	CQ 8
10	$\neg Fb$	
11	$Fb$	
12	$\perp$	$\neg E$ 11, 10
13	$Ga$	X 12
14	$Fb \rightarrow Ga$	$\rightarrow I$ 11–13
15	$\exists x(Fx \rightarrow Ga)$	$\exists I$ 14
16	$\exists x(Fx \rightarrow Ga)$	$\exists E$ 9, 10–15
17	$\exists x(Fx \rightarrow Ga)$	LEM 2–7, 8–16

NB: the variable 'x' does not occur in 'Ga'.

When all the quantifiers occur at the beginning of a sentence, that sentence is said to be in *prenex normal form*. Together with the CQ rules, these equivalences are sometimes called *prenexing rules*, since they give us a means for putting any sentence into prenex normal form.

CHAPTER 34

# *Rules for identity*

A. Provide a proof of each claim.

1.  $Pa \vee Qb, Qb \rightarrow b = c, \neg Pa \vdash Qc$

1	$Pa \vee Qb$	
2	$Qb \rightarrow b = c$	
3	$\neg Pa$	
4	$Qb$	DS 1, 3
5	$b = c$	$\rightarrow$ E 2, 4
6	$Qc$	$=$ E 5, 4

2.  $m = n \vee n = o, An \vdash Am \vee Ao$

1	$m = n \vee n = o$	
2	$An$	
3	$m = n$	
4	$Am$	$=$ E 3, 2
5	$Am \vee Ao$	$\vee$ I 4
6	$n = o$	
7	$Ao$	$=$ E 6, 7
8	$Am \vee Ao$	$\vee$ I 7
9	$Am \vee Ao$	$\vee$ E 1, 3-5, 6-8

3.  $\forall x x = m, Rma \vdash \exists x Rxx$

- |   |                   |               |
|---|-------------------|---------------|
| 1 | $\forall x x = m$ |               |
| 2 | $Rma$             |               |
| 3 | $a = m$           | $\forall E$ 1 |
| 4 | $Raa$             | $=E$ 3, 2     |
| 5 | $\exists x Rxx$   | $\exists I$ 4 |
4.  $\forall x \forall y (Rxy \rightarrow x = y) \vdash Rab \rightarrow Rba$
- |   |   |                      |
|---|---|----------------------|
| 1 | $\forall x \forall y (Rxy \rightarrow x = y)$ |                      |
| 2 | $Rab$   |                      |
| 3 | $\forall y (Ray \rightarrow a = y)$           | $\forall E$ 1        |
| 4 | $Rab \rightarrow a = b$                       | $\forall E$ 3        |
| 5 | $a = b$                                       | $\rightarrow E$ 4, 2 |
| 6 | $Raa$   | $=E$ 5, 2            |
| 7 | $Rba$   | $=E$ 5, 6            |
| 8 | $Rab \rightarrow Rba$                         | $\rightarrow I$ 2-7  |
5.  $\neg \exists x \neg x = m \vdash \forall x \forall y (Px \rightarrow Py)$
- |    |   |                     |
|----|---|---------------------|
| 1  | $\neg \exists x \neg x = m$               |                     |
| 2  | $\forall x \neg \neg x = m$               | CQ 1                |
| 3  | $\neg \neg a = m$                         | $\forall E$ 2       |
| 4  | $a = m$                                   | DNE 3               |
| 5  | $\neg \neg b = m$                         | $\forall E$ 2       |
| 6  | $b = m$                                   | DNE 5               |
| 7  | $Pa$                                      |                     |
| 8  | $Pm$                                      | $=E$ 3, 7           |
| 9  | $Pb$                                      | $=E$ 5, 8           |
| 10 | $Pa \rightarrow Pb$                       | $\rightarrow I$ 7-9 |
| 11 | $\forall y (Pa \rightarrow Py)$           | $\forall I$ 10      |
| 12 | $\forall x \forall y (Px \rightarrow Py)$ | $\forall I$ 11      |
6.  $\exists x Jx, \exists x \neg Jx \vdash \exists x \exists y \neg x = y$

1	$\exists x Jx$	
2	$\exists x \neg Jx$	
3	$J a$	
4	$\neg J b$	
5	$a = b$	
6	$J b$	$=E$ 5, 3
7	$\perp$	$\neg E$ 6, 4
8	$\neg a = b$	$\neg I$ 5-7
9	$\exists y \neg a = y$	$\exists I$ 8
10	$\exists x \exists y \neg x = y$	$\exists I$ 9
11	$\exists x \exists y \neg x = y$	$\exists E$ 2, 4-10
12	$\exists x \exists y \neg x = y$	$\exists E$ 1, 3-11
7. $\forall x(x = n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$		
1	$\forall x(x = n \leftrightarrow Mx)$	
2	$\forall x(Ox \vee \neg Mx)$	
3	$n = n \leftrightarrow Mn$	$\forall E$ 1
4	$n = n$	$=I$
5	$Mn$	$\leftrightarrow E$ 3, 4
6	$On \vee \neg Mn$	$\forall E$ 2
7	$\neg On$	
8	$\neg Mn$	$DS$ 6, 7
9	$\perp$	$\neg E$ 5, 8
10	$\neg \neg On$	$\neg I$ 7-9
11	$On$	$DNE$ 10
8. $\exists x Dx, \forall x(x = p \leftrightarrow Dx) \vdash Dp$		
1	$\exists x Dx$	
2	$\forall x(x = p \leftrightarrow Dx)$	
3	$Dc$	
4	$c = p \leftrightarrow Dc$	$\forall E$ 2
5	$c = p$	$\leftrightarrow E$ 4, 3
6	$Dp$	$=E$ 5, 3
7	$Dp$	$\exists E$ 1, 3-6

9.  $\exists x[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx], Kd \vdash Bd$

1	$\exists x[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx]$	
2	$Kd$	
3	$(Ka \wedge \forall y(Ky \rightarrow a = y)) \wedge Ba$	
4	$Ka \wedge \forall y(Ky \rightarrow a = y)$	$\wedge E$ 3
5	$Ka$	$\wedge E$ 4
6	$\forall y(Ky \rightarrow a = y)$	$\wedge E$ 4
7	$Kd \rightarrow a = d$	$\forall E$ 6
8	$a = d$	$\rightarrow E$ 7, 2
9	$Ba$	$\wedge E$ 3
10	$Bd$	$=E$ 8, 9
11	$Bd$	$\exists E$ 1, 3–10

10.  $\vdash Pa \rightarrow \forall x(Px \vee \neg x = a)$

1	$Pa$	
2	$b = a$	
3	$Pb$	$=E$ 2, 1
4	$Pb \vee \neg b = a$	$\vee I$ 3
5	$\neg b = a$	
6	$Pb \vee \neg b = a$	$\vee I$ 5
7	$Pb \vee \neg b = a$	LEM 2–4, 5–6
8	$\forall x(Px \vee \neg x = a)$	$\forall I$ 7
9	$Pa \rightarrow \forall x(Px \vee \neg x = a)$	$\rightarrow I$ 1–8

**B.** Show that the following are provably equivalent:

- $\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$
- $Fn \wedge \forall y(Fy \rightarrow n = y)$

And hence that both have a decent claim to symbolize the English sentence ‘Nick is the F’.

**In one direction:**

1	$\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$	
2	$[Fa \wedge \forall y(Fy \rightarrow a = y)] \wedge a = n$	
3	$a = n$	$\wedge E$ 2
4	$Fa \wedge \forall y(Fy \rightarrow a = y)$	$\wedge E$ 2
5	$Fa$	$\wedge E$ 4
6	$Fn$	$=E$ 3, 5
7	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 4
8	$\forall y(Fy \rightarrow n = y)$	$=E$ 3, 7
9	$Fn \wedge \forall y(Fy \rightarrow n = y)$	$\wedge I$ 6, 8
10	$Fn \wedge \forall y(Fy \rightarrow n = y)$	$\exists E$ 1, 2–9

And now in the other:

1	$Fn \wedge \forall y(Fy \rightarrow n = y)$	
2	$n = n$	$=I$
3	$[Fn \wedge \forall y(Fy \rightarrow n = y)] \wedge n = n$	$\wedge I$ 1, 2
4	$\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$	$\exists I$ 3

**C.** In §24, we claimed that the following are logically equivalent symbolizations of the English sentence ‘there is exactly one F’:

- $\exists xFx \wedge \forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$
- $\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$
- $\exists x\forall y(Fy \leftrightarrow x = y)$

Show that they are all provably equivalent. (*Hint:* to show that three claims are provably equivalent, it suffices to show that the first proves the second, the second proves the third and the third proves the first; think about why.)

It suffices to show that the first proves the second, the second proves the third and the third proves the first, for we can then show that any of them prove any others, just by chaining the proofs together (numbering lines, where necessary). Armed with this, we start on the first proof:

1	$\exists xFx \wedge \forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$	
2	$\exists xFx$	$\wedge E$ 1
3	$\forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$	$\wedge E$ 1
4	$Fa$	
5	$\forall y[(Fa \wedge Fy) \rightarrow a = y]$	$\forall E$ 3
6	$(Fa \wedge Fb) \rightarrow a = b$	$\forall E$ 5
7	$Fb$	
8	$Fa \wedge Fb$	$\wedge I$ 4, 7
9	$a = b$	$\rightarrow E$ 6, 8
10	$Fb \rightarrow a = b$	$\rightarrow I$ 7-9
11	$\forall y(Fy \rightarrow a = y)$	$\forall I$ 10
12	$Fa \wedge \forall y(Fy \rightarrow a = y)$	$\wedge I$ 4, 11
13	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	$\exists I$ 12
14	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	$\exists E$ 2, 4-13

Now for the second proof:

1	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	
2	$Fa \wedge \forall y(Fy \rightarrow a = y)$	
3	$Fa$	$\wedge E$ 2
4	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 2
5	$Fb$	
6	$Fb \rightarrow a = b$	$\forall E$ 4
7	$a = b$	$\rightarrow E$ 6, 5
8	$a = b$	
9	$Fb$	$=E$ 8, 3
10	$Fb \leftrightarrow a = b$	$\leftrightarrow I$ 5-7, 8-9
11	$\forall y(Fy \leftrightarrow a = y)$	$\forall I$ 10
12	$\exists x\forall y(Fy \leftrightarrow x = y)$	$\exists I$ 11
13	$\exists x\forall y(Fy \leftrightarrow x = y)$	$\exists E$ 1, 2-12

And finally, the third proof:



1	$\exists x \forall y (Fy \leftrightarrow x = y)$	
2	$\forall y (Fy \leftrightarrow a = y)$	
3	$Fa \leftrightarrow a = a$	VE 2
4	$a = a$	=I
5	$Fa$	$\leftrightarrow$ E 3, 4
6	$\exists x Fx$	$\exists$ I 5
7	$Fb \wedge Fc$	
8	$Fb$	$\wedge$ E 7
9	$Fb \leftrightarrow a = b$	VE 2
10	$a = b$	$\leftrightarrow$ E 9, 8
11	$Fc$	$\wedge$ E 7
12	$Fc \leftrightarrow a = c$	VE 2
13	$a = c$	$\leftrightarrow$ E 12, 11
14	$b = c$	=E 10, 13
15	$(Fb \wedge Fc) \rightarrow b = c$	$\rightarrow$ I 8–14
16	$\forall y [(Fb \wedge Fy) \rightarrow b = y]$	$\forall$ I 15
17	$\forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\forall$ I 16
18	$\exists x Fx \wedge \forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\wedge$ I 6, 17
19	$\exists x Fx \wedge \forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\exists$ E 1, 2–18

**D.** Symbolize the following argument

There is exactly one F. There is exactly one G. Nothing is both F and G. So: there are exactly two things that are either F or G.

And offer a proof of it.

1.  $\exists x [Fx \wedge \forall y (Fy \rightarrow x = y)]$
  2.  $\exists x [Gx \wedge \forall y (Gy \rightarrow x = y)]$
  3.  $\forall x (\neg Fx \vee \neg Gx) \therefore$
- $\therefore \exists x \exists y [\neg x = y \wedge \forall z ((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$

1	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	
2	$\exists x[Gx \wedge \forall y(Gy \rightarrow x = y)]$	
3	$\forall x(\neg Fx \vee \neg Gx)$	
4	<u><math>Fa \wedge \forall y(Fy \rightarrow a = y)</math></u>	
5	$Fa$	$\wedge E$ 4
6	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 4
7	$\neg Fa \vee \neg Ga$	$\vee E$ 3
8	$\neg Ga$	DS 7, 5
9	<u><math>Gb \wedge \forall y(Gy \rightarrow b = y)</math></u>	
10	$Gb$	$\wedge E$ 9
11	$\forall y(Gy \rightarrow b = y)$	$\wedge E$ 9
12	<u><math>a = b</math></u>	
13	$Ga$	=E 12, 10
14	$\perp$	$\neg E$ 13, 8
15	$\neg a = b$	$\neg I$ 12–14
16	<u><math>Fc \vee Gc</math></u>	
17	<u><math>Fc</math></u>	
18	$Fc \rightarrow a = c$	$\vee E$ 6
19	$a = c$	$\rightarrow E$ 18, 17
20	$a = c \vee b = c$	$\vee I$ 19
21	<u><math>Gc</math></u>	
22	$Gc \rightarrow b = c$	$\vee E$ 11
23	$b = c$	$\rightarrow E$ 22, 21
24	$a = c \vee b = c$	$\vee I$ 23
25	$a = c \vee b = c$	$\vee E$ 16, 17–20, 21–24
26	$(Fc \vee Gc) \rightarrow (a = c \vee b = c)$	$\rightarrow I$ 16–25
27	$\forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\forall I$ 26
28	$\neg a = b \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\wedge I$ 15, 27
29	$\exists y[\neg a = y \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee y = z))]$	$\exists I$ 28
30	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists I$ 29
31	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists E$ 2, 9–30
32	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists E$ 1, 4–31

## CHAPTER 35

# *Derived rules*

A. Offer proofs which justify the addition of the second and fourth CQ rules as derived rules.

Justification for the second rule:

1	$\neg\exists xAx$	
2	$Aa$	
3	$\exists xAx$	$\exists I$ 2
4	$\perp$	$\neg E$ 3, 1
5	$\neg Aa$	$\neg I$ 2-4
6	$\forall x\neg Ax$	$\forall I$ 5

The fourth rule is harder to justify. Here is a proof that is relatively straightforward, but uses the derived rule DNE:

1	$\neg\forall xAx$	
2	$\neg\exists x\neg Ax$	
3	$\neg Aa$	
4	$\exists x\neg Ax$	$\exists I$ 3
5	$\perp$	$\neg E$ 4, 2
6	$\neg\neg Aa$	$\neg I$ 3-5
7	$Aa$	DNE 6
8	$\forall xAx$	$\forall I$ 7
9	$\perp$	$\neg E$ 8, 1
10	$\neg\neg\exists x\neg Ax$	$\neg I$ 2-9
11	$\exists x\neg Ax$	DNE 10

And here is a proof that does not use any derived rules:

1	$\neg\forall xAx$	
2	$\exists x\neg Ax$	
3	$\exists x\neg Ax \wedge \exists x\neg Ax$	$\wedge I$ 2
4	$\exists x\neg Ax$	$\wedge E$ 3
5	$\neg\exists x\neg Ax$	
6	$Ab$	
7	$Ab \wedge Ab$	$\wedge I$ 6
8	$Ab$	$\wedge E$ 7
9	$\neg Ab$	
10	$\exists x\neg Ax$	$\exists I$ 9
11	$\perp$	$\neg E$ 10, 5
12	$Ab$	X 11
13	$Ab$	LEM 6–8, 9–12
14	$\forall xAx$	$\forall I$ 13
15	$\perp$	$\neg E$ 14, 1
16	$\exists x\neg Ax$	X 15
17	$\exists x\neg Ax$	LEM 2–4, 5–16

## CHAPTER 37

# *Normal Forms and Expressive Completeness*

### Practice exercises

A. Consider the following sentences:

1.  $(A \rightarrow \neg B)$
2.  $\neg(A \leftrightarrow B)$
3.  $(\neg A \vee \neg(A \wedge B))$
4.  $(\neg(A \rightarrow B) \wedge (A \rightarrow C))$
5.  $(\neg(A \vee B) \leftrightarrow ((\neg C \wedge \neg A) \rightarrow \neg B))$
6.  $((\neg(A \wedge \neg B) \rightarrow C) \wedge \neg(A \wedge D))$

For each sentence, find a tautologically equivalent sentence in DNF and one in CNF.