forall *x*

Calgary Remix

Solutions to Selected Exercises

By **P. D. Magnus**
Tim Button
with additions by
J. Robert Loftis
remixed and revised by
Aaron Thomas-Bolduc
Richard Zach

Summer 2017 *bis*

This booklet is based on the solutions booklet forall$x$: *Cambridge*, by

Tim Button
University of Cambridge

used under a CC BY-SA 3.0 license, which is based in turn on forall$x$, by

P.D. Magnus
University at Albany, State University of New York

used under a CC BY-SA 3.0 license,
which was remixed & expanded by

Aaron Thomas-Bolduc & Richard Zach
University of Calgary

It includes additional material from forall$x$ by P.D. Magnus, used under a CC BY-SA 3.0 license, and from forall$x$: *Lorain County Remix*, by Cathal Woods and J. Robert Loftis, used under a CC BY-SA 4.0 license.

The LaTeX source for this book is available on GitHub. This version is revision e491ff2 (2017-08-18).

The preparation of this textbook was made possible by a grant from the Taylor Institute for Teaching and Learning.

UNIVERSITY OF CALGARY
Taylor Institute for Teaching and Learning

Contents

CHAPTER 1

Arguments

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

CHAPTER 2

Valid arguments

A. Which of the following arguments is valid? Which is invalid?

1. Socrates is a man.
2. All men are carrots.

∴ Socrates is a carrot. Valid

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

∴ Abe Lincoln was born in Illinois. Valid

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

∴ Abe Lincoln will not die. Invalid

Abe Lincoln might die for some other reason: someone else might pull the trigger; he might die of old age.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.

∴ Abe Lincoln was from France. Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

∴ The world will not end today. Valid

1. Joe is now 19 years old.
2. Joe is now 87 years old.

∴ Bob is now 20 years old. Valid

An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. It is impossible for all the premises to be true; so it is certainly impossible that the premises are all true and the conclusion is false.

B. Could there be:

1. A valid argument that has one false premise and one true premise? Yes.
   Example: the first argument, above.
2. A valid argument that has only false premises? Yes.
   Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.
3. A valid argument with only false premises and a false conclusion? Yes.
   The same example will suffice.
4. An invalid argument that can be made valid by the addition of a new premise? Yes.
   Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
5. A valid argument that can be made invalid by the addition of a new premise? No.
   An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

CHAPTER 3

Other logical notions

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

B. For each of the following: Is it a necessary truth, a necessary falsehood, or contingent?

1. Elephants dissolve in water.
2. Wood is a light, durable substance useful for building things.
3. If wood were a good building material, it would be useful for building things.
4. I live in a three story building that is two stories tall.
5. If gerbils were mammals they would nurse their young.

C. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will disintegrate.
2. All mammals dissolve in water.
   If you put an elephant in water, it will disintegrate.
3. George Bush was the 43rd president.
   Barack Obama is the 44th president.

4. Barack Obama is the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.
   All mammals dissolve in water.

D. Which of the following pairs of sentences are necessarily equivalent?

1. Thelonious Monk played piano.
   John Coltrane played tenor sax.
2. Thelonious Monk played gigs with John Coltrane.
   John Coltrane played gigs with Thelonious Monk.
3. All professional piano players have big hands.
   Piano player Bud Powell had big hands.
4. Bud Powell suffered from severe mental illness.
   All piano players suffer from severe mental illness.
5. John Coltrane was deeply religious.
   John Coltrane viewed music as an expression of spirituality.

E. Consider the following sentences:

G1 There are at least four giraffes at the wild animal park.

G2 There are exactly seven gorillas at the wild animal park.

G3 There are not more than two Martians at the wild animal park.

G4 Every giraffe at the wild animal park is a Martian.

Now consider each of the following collections of sentences. Which are jointly possible? Which are jointly impossible?

1. Sentences G2, G3, and G4 — Jointly possible
2. Sentences G1, G3, and G4 — Jointly impossible
3. Sentences G1, G2, and G4 — Jointly possible
4. Sentences G1, G2, and G3 — Jointly possible

F. Consider the following sentences.

M1 All people are mortal.

M2 Socrates is a person.

M3 Socrates will never die.

M4 Socrates is mortal.

Which combinations of sentences are jointly possible? Mark each “possible” or “impossible.”

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4

3. Sentences M2 and M3
4. Sentences M1 and M4
5. Sentences M1, M2, M3, and M4

G. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument that has one false premise and one true premise
   Yes: 'All whales are mammals (*true*). All mammals are plants (*false*). So all whales are plants.'
2. A valid argument that has a false conclusion
   Yes. (See example from previous exercise.)
3. A valid argument, the conclusion of which is a necessary falsehood
   Yes: '$1 + 1 = 3$. So $1 + 2 = 4$.'
4. An invalid argument, the conclusion of which is a necessary truth
   No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
5. A necessary truth that is contingent
   No. If a sentence is a necessary truth, it cannot possibly be false, but a contingent sentence can be false.
6. Two necessarily equivalent sentences, both of which are necessary truths
   Yes: '4 is even', '4 is divisible by 2'.
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
   No. A necessary truth cannot possibly be false, while a contingent sentence can be false. So in any situation in which the contingent sentence is false, it will have a different truth value from the necessary truth. Thus they will not necessarily have the same truth value, and so will not be equivalent.
8. Two necessarily equivalent sentences that together are jointly impossible
   Yes: '$1 + 1 = 4$' and '$1 + 1 = 3$'.
9. A jointly possible collection of sentences that contains a necessary falsehood
   No. If a sentence is necessarily false, there is no way to make it true, let alone it along with all the other sentences.
10. A jointly impossible set of sentences that contains a necessary truth
    Yes: '$1 + 1 = 4$' and '$1 + 1 = 2$'.

H. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
2. A valid argument with true premises and a false conclusion
3. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
4. A jointly possible collection of sentences, all of which are contingent
5. A false necessary truth

6. A valid argument with false premises
7. A necessarily equivalent pair of sentences that are not jointly possible
8. A necessary truth that is also a necessary falsehood
9. A jointly possible collection of sentences that are all necessary falsehoods

CHAPTER 5

Connectives

A. Using the symbolization key given, symbolize each English sentence in TFL.

M: Those creatures are men in suits.
C: Those creatures are chimpanzees.
G: Those creatures are gorillas.

1. Those creatures are not men in suits.
   $\neg M$
2. Those creatures are men in suits, or they are not.
   $(M \vee \neg M)$
3. Those creatures are either gorillas or chimpanzees.
   $(G \vee C)$
4. Those creatures are neither gorillas nor chimpanzees.
   $\neg(C \vee G)$
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
   $(C \rightarrow \neg(G \vee M))$
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.
   $(M \vee (C \vee G))$

B. Using the symbolization key given, symbolize each English sentence in TFL.

A: Mister Ace was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mister Edge was murdered.
F: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
   $(A \vee E)$
2. If Mister Ace was murdered, then the cook did it.
   $(A \rightarrow C)$

3. If Mister Edge was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
   $(B \vee D)$
5. The cook did it only if the Duchess is lying.
   $(C \rightarrow D)$
6. If the murder weapon was a frying pan, then the culprit must have been the cook.
   $(F \rightarrow C)$
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
   $(\neg F \rightarrow (C \vee B))$
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
   $(A \leftrightarrow \neg E)$
9. The Duchess is lying, unless it was Mister Edge who was murdered.
   $(D \vee E)$
10. If Mister Ace was murdered, he was done in with a frying pan.
    $(A \rightarrow F)$
11. Since the cook did it, the butler did not.
    $(C \wedge \neg B)$
12. Of course the Duchess is lying!
    D

C. Using the symbolization key given, symbolize each English sentence in TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
   $(E_1 \wedge E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.
   $(F_1 \rightarrow S_1)$
3. Ava is a firefighter, unless she is an electrician.
   $(F_1 \vee E_1)$
4. Harrison is an unsatisfied electrician.
   $(E_2 \wedge \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
   $((E_1 \wedge E_2) \wedge \neg(S_1 \vee S_2))$
7. Harrison is satisfied only if he is a firefighter.

$(S_2 \to F_2)$

8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $((\neg E_1 \to \neg E_2) \wedge (E_1 \to E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
    $((E_2 \wedge F_2) \to S_2)$
11. It cannot be that Harrison is both an electrician and a firefighter.
    $\neg(E_2 \wedge F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
    $((F_2 \wedge F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D. Using the symbolization key given, translate each English-language sentence into TFL.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
   $J_1 \wedge J_2$
2. Neither Miles Davis nor John Coltrane played tuba.
   $\neg(M_2 \vee J_3)$ or $\neg M_2 \wedge \neg J_3$
3. John Coltrane did not play both tenor sax and tuba.
   $\neg(J_1 \wedge J_3)$ or $\neg J_1 \vee \neg J_3$
4. John Coltrane did not play tenor sax unless he also played soprano sax.
   $\neg J_1 \vee J_2$
5. John Coltrane did not play tuba, but Miles Davis did.
   $\neg J_3 \wedge M_2$
6. Miles Davis played trumpet only if he also played tuba.
   $M_1 \to M_2$
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
   $M_1 \to (J_1 \vee (J_2 \vee J_3))$
8. If John Coltrane played tuba then Miles Davis played neither trumpet nor tuba.
   $J_3 \to \neg(M_1 \vee M_2)$ or $J_3 \to (\neg M_1 \wedge \neg M_2)$
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.
   $(J_3 \wedge M_2) \leftrightarrow (\neg J_1 \wedge \neg M_1)$ or $(J_3 \wedge M_2) \leftrightarrow \neg(J_1 \vee M_1)$

F. Give a symbolization key and symbolize the following English sentences in TFL.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
G: The German embassy will be in an uproar.

1. Alice and Bob are both spies.
   $(A \wedge B)$
2. If either Alice or Bob is a spy, then the code has been broken.
   $((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
   $(\neg(A \vee B) \rightarrow \neg C)$
4. The German embassy will be in an uproar, unless someone has broken the code.
   $(G \vee C)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
   $((C \vee \neg C) \wedge G)$
6. Either Alice or Bob is a spy, but not both.
   $((A \vee B) \wedge \neg(A \wedge B))$

G. Give a symbolization key and symbolize the following English sentences in TFL.

F: There is food to be found in the pridelands.
R: Rafiki will talk about squashed bananas.
A: Simba is alive.
K: Scar will remain as king.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
   $(F \rightarrow R)$
2. Rafiki will talk about squashed bananas unless Simba is alive.
   $(R \vee A)$
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
   $((R \vee \neg R) \wedge F)$
4. Scar will remain as king if and only if there is food to be found in the pridelands.
   $(K \leftrightarrow F)$
5. If Simba is alive, then Scar will not remain as king.
   $(A \rightarrow \neg K)$

H. For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

    P: Dorothy plays the Piano in the morning.
    C: Roger wakes up cranky.
    D: Dorothy is distracted.

    $(P \to C)$, $(P \lor D)$, $(\neg C \to D)$
2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

    T_1: It rains on Tuesday
    T_2: It snows on Tuesday
    S: Neville is sad on Tuesday
    C: Neville is cold on Tuesday

    $(T_1 \lor T_2)$, $(T_1 \to S)$, $(T_2 \to C)$, $(S \lor C)$
3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

    Z: Zoog remembered to do his chores
    C: Things are clean
    N: Things are neat

    $(Z \to (C \land \neg N))$, $(\neg Z \to (N \land \neg C))$, $((N \lor C) \land \neg(N \land C))$.

I. For each argument, write a symbolization key and translate the argument as well as possible into TFL. The part of the passage in italics is there to provide context for the argument, and doesn't need to be symbolized.

1. It is going to rain soon. I know because my leg is hurting, and my leg hurts if it's going to rain.
2. *Spider-man tries to figure out the bad guy's plan.* If Doctor Octopus gets the uranium, he will blackmail the city. I am certain of this because if Doctor Octopus gets the uranium, he can make a dirty bomb, and if he can make a dirty bomb, he will blackmail the city.
3. *A westerner tries to predict the policies of the Chinese government.* If the Chinese government cannot solve the water shortages in Beijing, they will have to move the capital. They don't want to move the capital. Therefore they must solve the water shortage. But the only way to solve the water shortage is to divert almost all the water from the Yangzi river northward. Therefore the Chinese government will go with the project to divert water from the south to the north.

J. We symbolized an *exclusive or* using '$\lor$', '$\land$', and '$\neg$'. How could you symbolize an *exclusive or* using only two connectives? Is there any way to symbolize an *exclusive or* using only one connective?
For two connectives, we could offer any of the following:

$$\neg(\mathscr{A} \leftrightarrow \mathscr{B})$$
$$(\neg\mathscr{A} \leftrightarrow \mathscr{B})$$
$$(\neg(\neg\mathscr{A} \wedge \neg\mathscr{B}) \wedge \neg(\mathscr{A} \wedge \mathscr{B}))$$

But if we wanted to symbolize it using only one connective, we would have to introduce a new primitive connective.

CHAPTER 6

Sentences of TFL

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes
4. $\neg \wedge S$ — (a) no (b) no
5. $(G \wedge \neg G)$ — (a) yes (b) yes
6. $(A \rightarrow (A \wedge \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes
7. $[(Z \leftrightarrow S) \rightarrow W] \wedge [J \vee X]$ — (a) no (b) yes
8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \wedge D)$ — (a) no (b) no

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.
No. Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, . . . , that were ultimately built out of atomic sentences.

C. What is the scope of each connective in the sentence

$$\big[(H \rightarrow I) \vee (I \rightarrow H)\big] \wedge (J \vee K)$$

The scope of the left-most instance of '$\rightarrow$' is '$(H \rightarrow I)$'.
The scope of the right-most instance of '$\rightarrow$' is '$(I \rightarrow H)$'.
The scope of the left-most instance of '$\vee$ is '$\big[(H \rightarrow I) \vee (I \rightarrow H)\big]$'
The scope of the right-most instance of '$\vee$' is '$(J \vee K)$'
The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

CHAPTER 10
Complete truth tables

A. Complete truth tables for each of the following:

1. $A \rightarrow A$

A	$A \rightarrow A$
T	T **T** T
F	F **T** F

2. $C \rightarrow \neg C$

C	$C \rightarrow \neg C$
T	T **F** F T
F	F **T** T F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
T	T	T T T **T** T T F F T
T	F	T F F **T** F T T T F
F	T	F F T **T** F F T F T
F	F	F T F **T** T F F T F

4. $(A \rightarrow B) \vee (B \rightarrow A)$

A	B	$(A \rightarrow B) \vee (B \rightarrow A)$
T	T	T T T **T** T T T
T	F	T F F **T** F T T
F	T	F T T **T** T F F
F	F	F T F **T** F T F

5. $(A \wedge B) \rightarrow (B \vee A)$

A	B	$(A \wedge B) \rightarrow (B \vee A)$
T	T	T T T **T** T T T
T	F	T F F **T** F T T
F	T	F F T **T** T T F
F	F	F F F **T** F F F

6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

A	B	$\neg (A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
T	T	F T T T **T** F T F F T
T	F	F T T F **T** F T F T F
F	T	F F T T **T** T F F F T
F	F	T F F F **T** T F T T F

7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$

A	B	C	$\big[(A \wedge B) \wedge \neg (A \wedge B)\big] \wedge C$
T	T	T	T T T F F T T T **F** T
T	T	F	T T T F F T T T **F** F
T	F	T	T F F F T T F F **F** T
T	F	F	T F F F T T F F **F** F
F	T	T	F F T F T F F F **F** T
F	T	F	F F T F T F F T **F** F
F	F	T	F F F F T F F T **F** T
F	F	F	F F F F T F F F **F** F

8. $[(A \wedge B) \wedge C] \rightarrow B$

A	B	C	$[(A \wedge B) \wedge C] \rightarrow B$
T	T	T	T T T T T **T** T
T	T	F	T T T F F **T** T
T	F	T	T F F F T **T** F
T	F	F	T F F F F **T** F
F	T	T	F F T F T **T** T
F	T	F	F F T F F **T** T
F	F	T	F F F F T **T** F
F	F	F	F F F F F **T** F

9. $\neg\big[(C \vee A) \vee B\big]$

A	B	C	$\neg[(C \vee A) \vee B]$
T	T	T	**F** T T T T T
T	T	F	**F** F T T T T
T	F	T	**F** T T T T F
T	F	F	**F** F T T T F
F	T	T	**F** T T F T T
F	T	F	**F** F F F T T
F	F	T	**F** T T F T F
F	F	F	**T** F F F F F

B. Check all the claims made in introducing the new notational conventions in §10.3, i.e. show that:

1. '$((A \wedge B) \wedge C)$' and '$(A \wedge (B \wedge C))$' have the same truth table

A	B	C	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **F** F	T **F** F F F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table

A	B	C	$(A \vee B) \vee C$	$A \vee (B \vee C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **T** F	T **T** T T F
T	F	T	T T F **T** T	T **T** F T T
T	F	F	T T F **T** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **T** F	F **T** T T F
F	F	T	F F F **T** T	F **T** F T T
F	F	F	F F F **F** F	F **F** F F F

3. '$((A \vee B) \wedge C)$' and '$(A \vee (B \wedge C))$' do not have the same truth table

A	B	C	$(A \vee B) \wedge C$	$A \vee (B \wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **T** T F F
T	F	T	T T F **T** T	T **T** F F T
T	F	F	T T F **F** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

4. '$((A \to B) \to C)$' and '$(A \to (B \to C))$' do not have the same truth table

A	B	C	$(A \to B) \to C$	$A \to (B \to C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **T** T	T **T** F T T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F T T
F	F	F	F T F **F** F	F **T** F T F

Also, check whether:

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table
Indeed they do:

A	B	C	$(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **T** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F F T
F	F	F	F T F **F** F	F **F** F T F

C. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $\neg(S \leftrightarrow (P \to S))$

$\boxed{\neg}$	(S	↔	(P	→	S))
F	T	T	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F
T	F	F	F	T	F

2. $\neg[(X \wedge Y) \vee (X \vee Y)]$

$\boxed{\neg}$	[(X	∧	Y)	∨	(X	∨	Y)]
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	F
F	F	F	T	T	F	T	T
T	F	F	F	F	F	F	F

3. $(A \to B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

(A	$\to$	B)	$\leftrightarrow$	($\neg$	B	$\leftrightarrow$	$\neg$	A)
T	T	T	T	F	T	T	F	T
T	F	F	T	T	F	F	F	T
F	T	T	F	F	T	F	T	F
F	T	F	T	T	F	T	T	F

4. $[C \leftrightarrow (D \lor E)] \land \neg C$

[C	$\leftrightarrow$	(D	$\lor$	E)]	$\land$	$\neg$	C
T	T	T	T	T	F	F	T
T	T	T	T	F	F	F	T
T	T	F	T	T	F	F	T
T	F	F	F	F	F	F	T
F	F	T	T	T	F	T	F
F	F	T	T	F	F	T	F
F	F	F	T	T	F	T	F
F	T	F	F	F	T	T	F

5. $\neg(G \land (B \land H)) \leftrightarrow (G \lor (B \lor H))$

$\neg$	(G	$\land$	(B	$\land$	H))	$\leftrightarrow$	(G	$\lor$	(B	$\lor$	H))
F	T	T	T	T	T	F	T	T	T	T	T
T	T	F	T	F	F	T	T	T	T	T	F
T	T	F	F	F	T	T	T	T	F	T	T
T	T	F	F	F	F	T	T	T	F	F	F
T	F	F	T	T	T	T	F	T	T	T	T
T	F	F	T	F	F	T	F	T	T	T	F
T	F	F	F	F	T	T	F	T	F	T	T
T	F	F	F	F	F	F	F	F	F	F	F

D. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $(D \land \neg D) \to G$

(D	$\land$	$\neg$	D)	$\to$	G
T	F	F	T	T	T
T	F	F	T	T	F
F	F	T	F	T	T
F	F	T	F	T	F

2. $(\neg P \vee \neg M) \leftrightarrow M$

(¬	P	∨	¬	M)	↔	M
F	T	F	F	T	T	T
F	T	T	T	F	F	F
T	F	T	F	T	T	T
T	F	T	T	F	T	F

3. $\neg\neg(\neg A \wedge \neg B)$

¬	¬	(¬	A	∧	¬	B)
F	T	F	T	F	F	T
F	T	F	T	F	T	F
F	T	T	F	F	F	T
T	F	T	F	T	T	F

4. $[(D \wedge R) \rightarrow I] \rightarrow \neg(D \vee R)$

[(D	∧	R)	→	I]	→	¬	(D	∨	R)
T	T	T	T	T	F	F	T	T	T
T	T	T	F	F	T	F	T	T	T
T	F	F	T	T	F	F	T	T	F
T	F	F	T	F	F	F	T	T	F
F	F	T	T	T	F	F	F	T	T
F	F	T	T	F	F	F	F	T	T
F	F	F	T	T	T	T	F	F	F
F	F	F	T	F	T	T	F	F	F

5. $\neg[(D \leftrightarrow O) \leftrightarrow A] \rightarrow (\neg D \wedge O)$

¬	[(D	↔	O)	↔	A]	→	(¬	D	∧	O)
F	T	T	T	T	T	T	F	T	F	T
T	T	T	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	F	T	F	F
F	T	F	F	T	F	T	F	T	F	F
T	F	F	T	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T	F	T	T
F	F	T	F	T	T	T	T	F	F	F
T	F	T	F	F	F	T	T	F	F	F

If you want additional practice, you can construct truth tables for any of the sentences and arguments in the exercises for the previous chapter.

CHAPTER 11

Semantic concepts

A. Revisit your answers to §10**A**. Determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

1. $A \rightarrow A$ — Tautology
2. $C \rightarrow \neg C$ — Neither
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ — Tautology
4. $(A \rightarrow B) \vee (B \rightarrow A)$ — Tautology
5. $(A \wedge B) \rightarrow (B \vee A)$ — Tautology
6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ — Tautology
7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$ — Contradiction
8. $[(A \wedge B) \wedge C] \rightarrow B$ — Tautology
9. $\neg\big[(C \vee A) \vee B\big]$ — Neither

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \rightarrow A$, $\neg A \rightarrow \neg A$, $A \wedge A$, $A \vee A$ — Jointly consistent (see line 1)

A	$A \rightarrow A$	$\neg A \rightarrow \neg A$	$A \wedge A$	$A \vee A$
T	T **T** T	F T **T** F T	T **T** T	T **T** T
F	F **T** F	T F **T** T F	F **F** F	F **F** F

2. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$ — Jointly consistent (see line 1)

A	B	C	$A \vee B$	$A \to C$	$B \to C$
T	T	T	T **T** T	T **T** T	T **T** T
T	T	F	T **T** T	T **F** F	T **F** F
T	F	T	T **T** T	T **T** T	F **T** T
T	F	F	T **T** F	T **F** F	F **T** F
F	T	T	F **T** F	F **T** T	T **T** T
F	T	F	F **T** T	F **T** F	T **F** F
F	F	T	F **F** F	F **T** T	F **T** T
F	F	F	F **F** F	F **T** F	F **T** F

3. $B \wedge (C \vee A)$, $A \to B$, $\neg(B \vee C)$ — Jointly inconsistent

A	B	C	$B \wedge (C \vee A)$	$A \to B$	$\neg (B \vee C)$
T	T	T	T **T** T T T	T **T** T	**F** T T T
T	T	F	T **T** F T T	T **T** T	**F** T T F
T	F	T	F **F** T T T	T **F** F	**F** F T T
T	F	F	F **F** F T T	T **F** F	**T** F F F
F	T	T	T **T** T T F	F **T** T	**F** T T T
F	T	F	T **F** F F F	F **T** T	**F** T T F
F	F	T	F **F** T T F	F **T** F	**F** F T T
F	F	F	F **F** F F F	F **T** F	**T** F F F

4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$ — Jointly consistent (see line 8)

A	B	C	$A \leftrightarrow (B \vee C)$	$C \to \neg A$	$A \to \neg B$
T	T	T	T **T** T T T	T **F** F T	T **F** F T
T	T	F	T **T** T T F	F **T** F T	T **F** F T
T	F	T	T **T** F T T	T **F** F T	T **T** T F
T	F	F	T **F** F F F	F **T** F T	T **T** T F
F	T	T	F **F** T T T	T **T** T F	F **T** F T
F	T	F	F **F** T T F	F **T** T F	F **T** F T
F	F	T	F **F** F T T	T **T** T F	F **T** T F
F	F	F	F **T** F F F	F **T** T F	F **T** T F

C. Use truth tables to determine whether each argument is valid or invalid.

1. $A \to A \therefore A$ — Invalid (see line 2)

A	$A \to A$	A
T	T **T** T	T
F	F **T** F	F

2. $A \to (A \wedge \neg A) \therefore \neg A$ — Valid

A	$A \to (A \wedge \neg A)$	$\neg A$
T	T **F** T F F T	**F** T
F	F **T** F F T F	**T** F

3. $A \vee (B \to A) \therefore \neg A \to \neg B$ — Valid

A	B	$A \vee (B \to A)$	$\neg A \to \neg B$
T	T	T **T** T T T	F T **T** F T
T	F	T **T** F T T	F T **T** T F
F	T	F **F** T F F	T F **F** F T
F	F	F **T** F T F	T F **T** T F

4. $A \vee B, B \vee C, \neg A \therefore B \wedge C$ Invalid (see line 6)

A	B	C	$A \vee B$	$B \vee C$	$\neg A$	$B \wedge C$
T	T	T	T **T** T	T **T** T	**F** T	T **T** T
T	T	F	T **T** T	T **T** F	**F** T	T **F** F
T	F	T	T **T** F	F **T** T	**F** T	F **F** T
T	F	F	T **T** F	F **F** F	**F** T	F **F** F
T	T	T	F **T** T	T **T** T	**T** F	T **T** T
T	T	F	F **T** T	T **T** F	**T** F	T **F** F
T	F	T	F **F** F	F **T** T	**T** F	F **F** T
T	F	F	F **F** F	F **F** F	**T** F	F **F** F

5. $(B \wedge A) \to C, (C \wedge A) \to B \therefore (C \wedge B) \to A$ Invalid (see line 5)

A	B	C	$(B \wedge A) \to C$	$(C \wedge A) \to B$	$(C \wedge B) \to A$
T	T	T	T T T **T** T	T T T **T** T	T T T **T** T
T	T	F	T T T **F** F	F F T **T** T	F F T **T** T
T	F	T	F F T **T** T	T T T **F** F	T F F **T** T
T	F	F	F F T **T** F	F F T **T** F	F F F **T** T
F	T	T	T F F **T** T	T F F **T** T	T T T **F** F
F	T	F	T F F **T** F	F F F **T** T	F F T **T** F
F	F	T	F F F **T** T	T F F **T** F	T F F **T** F
F	F	F	F F F **T** F	F F F **T** F	F F F **T** F

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence, using a complete truth table.

1. $\neg B \wedge B$ Contradiction
2. $\neg D \vee D$ Tautology
3. $(A \wedge B) \vee (B \wedge A)$ Contingent
4. $\neg[A \to (B \to A)]$ Contradiction
5. $A \leftrightarrow [A \to (B \wedge \neg B)]$ Contradiction
6. $[(A \wedge B) \leftrightarrow B] \to (A \to B)$ Contingent

E. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "Not equivalent."

1. A and $\neg A$
2. $A \wedge \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$

4. $A \vee (B \wedge C)$ and $(A \vee B) \wedge (A \vee C)$
5. $[A \wedge (A \vee B)] \rightarrow B$ and $A \rightarrow B$

F. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are equivalent, write "equivalent." Otherwise write, "not equivalent."

1. $A \rightarrow A$ and $A \leftrightarrow A$
2. $\neg(A \rightarrow B)$ and $\neg A \rightarrow \neg B$
3. $A \vee B$ and $\neg A \rightarrow B$
4. $(A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$
5. $A \leftrightarrow (B \leftrightarrow C)$ and $A \wedge (B \wedge C)$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent using a complete truth table.

1. $A \wedge \neg B$, $\neg(A \rightarrow B)$, $B \rightarrow A$

A	∧	¬	B	¬	(A	→	B)	B	→	A
T	F	F	T	F	T	T	T	T	T	T
T	**T**	T	F	**T**	T	F	F	F	**T**	T
F	F	F	T	F	F	T	T	T	F	F
F	F	T	F	F	F	T	F	F	T	F

Consistent

2. $A \vee B$, $A \rightarrow \neg A$, $B \rightarrow \neg B$

A	∨	B	A	→	¬	A	B	→	¬	B
T	T	T	T	F	F	T	T	F	F	T
T	T	F	T	F	F	T	F	T	T	F
F	T	T	F	T	T	F	T	F	F	T
F	F	F	F	T	T	F	F	T	T	F

Inconsistent

3. $\neg(\neg A \vee B)$, $A \rightarrow \neg C$, $A \rightarrow (B \rightarrow C)$

Consistent

¬	(¬	A	∨	B)	A	→	¬	C	A	→	(B	→	C)
F	F	T	T	T	T	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	F	T	F	T	F	F
T	F	T	F	F	T	F	F	T	T	T	F	T	T
T	F	T	F	F	T	T	T	F	T	T	F	T	F
F	T	F	T	T	F	T	F	T	F	F	T	T	T
F	T	F	T	T	F	T	T	F	F	T	T	F	F
F	T	F	T	F	F	T	F	T	F	T	F	T	T
F	T	F	T	F	F	T	T	F	F	T	F	T	F

4. $A \to B$, $A \wedge \neg B$ Inconsistent
5. $A \to (B \to C)$, $(A \to B) \to C$, $A \to C$ Consistent

H. Determine whether each collection of sentences is jointly consistent or jointly inconsistent, using a complete truth table.

1. $\neg B$, $A \to B$, A Inconsistent
2. $\neg(A \vee B)$, $A \leftrightarrow B$, $B \to A$ Consistent
3. $A \vee B$, $\neg B$, $\neg B \to \neg A$ Inconsistent
4. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \to B$ Consistent
5. $(A \vee B) \vee C$, $\neg A \vee \neg B$, $\neg C \vee \neg B$ Consistent

I. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \to B$, $B \therefore A$ Invalid
2. $A \leftrightarrow B$, $B \leftrightarrow C \therefore A \leftrightarrow C$ Valid
3. $A \to B$, $A \to C \therefore B \to C$ Invalid.
4. $A \to B$, $B \to A \therefore A \leftrightarrow B$ Valid

J. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \vee \big[A \to (A \leftrightarrow A)\big] \therefore A$ Invalid
2. $A \vee B$, $B \vee C$, $\neg B \therefore A \wedge C$ Valid
3. $A \to B$, $\neg A \therefore \neg B$ Invalid
4. A, $B \therefore \neg(A \to \neg B)$ Valid
5. $\neg(A \wedge B)$, $A \vee B$, $A \leftrightarrow B \therefore C$ Valid

K. Answer each of the questions below and justify your answer.

1. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent. What can you say about $\mathscr{A} \leftrightarrow \mathscr{B}$?
   $\mathscr{A}$ and $\mathscr{B}$ have the same truth value on every line of a complete truth table, so $\mathscr{A} \leftrightarrow \mathscr{B}$ is true on every line. It is a tautology.
2. Suppose that $(\mathscr{A} \wedge \mathscr{B}) \to \mathscr{C}$ is neither a tautology nor a contradiction. What can you say about whether $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid?
   Since the sentence $(\mathscr{A} \wedge \mathscr{B}) \to \mathscr{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, $\mathscr{A}$ and $\mathscr{B}$ are true and $\mathscr{C}$ is false. So the argument is invalid.
3. Suppose that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are jointly inconsistent. What can you say about $(\mathscr{A} \wedge \mathscr{B} \wedge \mathscr{C})$?
   Since the sentences are jointly inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a contradiction
4. Suppose that $\mathscr{A}$ is a contradiction. What can you say about whether $\mathscr{A}, \mathscr{B} \vDash \mathscr{C}$?

Since $\mathscr{A}$ is false on every line of a complete truth table, there is no line on which $\mathscr{A}$ and $\mathscr{B}$ are true and $\mathscr{C}$ is false. So the entailment holds.

5. Suppose that $\mathscr{C}$ is a tautology. What can you say about whether $\mathscr{A}, \mathscr{B} \vDash \mathscr{C}$?
Since $\mathscr{C}$ is true on every line of a complete truth table, there is no line on which $\mathscr{A}$ and $\mathscr{B}$ are true and $\mathscr{C}$ is false. So the entailment holds.
6. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent. What can you say about $(\mathscr{A} \vee \mathscr{B})$?
Not much. Since $\mathscr{A}$ and $\mathscr{B}$ are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is logically equivalent to them.
7. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are *not* logically equivalent. What can you say about $(\mathscr{A} \vee \mathscr{B})$?
$\mathscr{A}$ and $\mathscr{B}$ have different truth values on at least one line of a complete truth table, and $(\mathscr{A} \vee \mathscr{B})$ will be true on that line. On other lines, it might be true or false. So $(\mathscr{A} \vee \mathscr{B})$ is either a tautology or it is contingent; it is *not* a contradiction.

L. Consider the following principle:

- Suppose $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent. Suppose an argument contains $\mathscr{A}$ (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced $\mathscr{A}$ with $\mathscr{B}$.

Is this principle correct? Explain your answer.
The principle is correct. Since $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent, they have the same truth table. So every valuation that makes $\mathscr{A}$ true also makes $\mathscr{B}$ true, and every valuation that makes $\mathscr{A}$ false also makes $\mathscr{B}$ false. So if no valuation makes all the premises true and the conclusion false, when $\mathscr{A}$ was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace $\mathscr{A}$ with $\mathscr{B}$.

CHAPTER 12

Truth table shortcuts

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1. $\neg B \wedge B$ — Contradiction

B	$\neg$	B	$\wedge$	B
T	F		**F**	
F			**F**	

2. $\neg D \vee D$ — Tautology

D	$\neg$	D	$\vee$	D
T			**T**	
F	T		**T**	

3. $(A \wedge B) \vee (B \wedge A)$ — Neither

A	B	$(A \wedge B)$	$\vee$	$(B \wedge A)$
T	T	T	**T**	
T	F	F	**F**	F
F	T	F	**F**	F
F	F	F	**F**	F

4. $\neg[A \rightarrow (B \rightarrow A)]$ — Contradiction

A	B	$\neg$	$[A \rightarrow$	$(B \rightarrow A)]$
T	T	**F**	T	T
T	F	**F**	T	T
F	T	**F**	T	
F	F	**F**	T	

5. $A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$ — Contradiction

A	B	$A \leftrightarrow$	$[A \rightarrow$	$(B \wedge$	$\neg B)]$
T	T	**F**	F	F	F
T	F	**F**	F	F	
F	T	**F**	T		
F	F	**F**	T		

6. $\neg(A \wedge B) \leftrightarrow A$ Neither

A	B	$\neg$	$(A \wedge B)$	$\leftrightarrow A$
T	T	F	T	**F**
T	F	T	F	**T**
F	T	T	F	**F**
F	F	T	F	**F**

7. $A \rightarrow (B \vee C)$ Neither

A	B	C	$A \rightarrow$	$(B \vee C)$
T	T	T	**T**	T
T	T	F	**T**	T
T	F	T	**T**	T
T	F	F	**F**	F
F	T	T	**T**	
F	T	F	**T**	
F	F	T	**T**	
F	F	F	**T**	

8. $(A \wedge \neg A) \rightarrow (B \vee C)$ Tautology

A	B	C	$(A \wedge$	$\neg A)$	$\rightarrow (B \vee C)$
T	T	T	**F**	F	T
T	T	F	**F**	F	T
T	F	T	**F**	F	T
T	F	F	**F**	F	T
F	T	T	**F**		T
F	T	F	**F**		T
F	F	T	**F**		T
F	F	F	**F**		T

9. $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$ Neither

A	B	C	D	$(B \wedge D)$	$\leftrightarrow$	$[A \leftrightarrow$	$(A \vee C)]$
T	T	T	T	T	**T**	T	T
T	T	T	F	F	**F**	T	T
T	T	F	T	T	**T**	T	T
T	T	F	F	F	**F**	T	T
T	F	T	T	F	**F**	T	T
T	F	T	F	F	**F**	T	T
T	F	F	T	F	**F**	T	T
T	F	F	F	F	**F**	T	T
F	T	T	T	T	**F**	F	T
F	T	T	F	F	**T**	F	T
F	T	F	T	T	**T**	T	F
F	T	F	F	F	**F**	T	F
F	F	T	T	F	**T**	F	T
F	F	T	F	F	**T**	F	T
F	F	F	T	F	**F**	T	F
F	F	F	F	F	**F**	T	F

CHAPTER 13

Partial truth tables

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are logically equivalent:

1. A, $\neg A$ — Not logically equivalent

A	A	$\neg A$
T	T	F

2. A, $A \vee A$ — Logically equivalent

A	A	$A \vee A$
T	T	T
T	T	T

3. $A \rightarrow A$, $A \leftrightarrow A$ — Logically equivalent

A	$A \rightarrow A$	$A \leftrightarrow A$
T	T	T
F	T	T

4. $A \vee \neg B$, $A \rightarrow B$ — Not logically equivalent

A	B	$A \vee \neg B$	$A \rightarrow B$
T	F	T	F

5. $A \wedge \neg A$, $\neg B \leftrightarrow B$ — Logically equivalent

A	B	$A \wedge \neg A$	$\neg B \leftrightarrow B$
T	T	**F** F	F **F**
T	F	**F** F	T **F**
F	T	**F**	F **F**
F	F	**F**	T **F**

6. $\neg(A \wedge B)$, $\neg A \vee \neg B$ Logically equivalent

A	B	$\neg\,(A \wedge B)$	$\neg A \vee \neg B$
T	T	**F** T	F **F** F
T	F	**T** F	F **T** T
F	T	**T** F	T **T** F
F	F	**T** F	T **T** T

7. $\neg(A \rightarrow B)$, $\neg A \rightarrow \neg B$ Not logically equivalent

A	B	$\neg\,(A \rightarrow B)$	$\neg A \rightarrow \neg B$
T	T	**F** T	F **T** F

8. $(A \rightarrow B)$, $(\neg B \rightarrow \neg A)$ Logically equivalent

A	B	$(A \rightarrow B)$	$(\neg B \rightarrow \neg A)$
T	T	T	F **T**
T	F	F	T **F** F
F	T	T	F **T**
F	F	T	T **T** T

B. Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \wedge B$, $C \rightarrow \neg B$, C Jointly inconsistent

A	B	C	$A \wedge B$	$C \rightarrow \neg B$	C
T	T	T	T	**F** F	T
T	T	F	T	**T**	F
T	F	T	F	**T** T	T
T	F	F	F	**T**	F
F	T	T	F	**F** F	T
F	T	F	F	**T**	F
F	F	T	F	**T** T	T
F	F	F	F	**T**	F

2. $A \rightarrow B$, $B \rightarrow C$, A, $\neg C$ Jointly inconsistent

A	B	C	$A \rightarrow B$	$B \rightarrow C$	A	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

3. $A \vee B$, $B \vee C$, $C \rightarrow \neg A$ Jointly consistent

A	B	C	$A \vee B$	$B \vee C$	$C \rightarrow \neg A$
F	T	T	T	T	**T** T

4. A, B, C, $\neg D$, $\neg E$, F — Jointly consistent

A	B	C	D	E	F	A	B	C	$\neg D$	$\neg E$	F
T	T	T	F	F	T	T	T	T	T	T	T

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

1. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \therefore A$ — Invalid

A	$A \vee \big[A \rightarrow (A \leftrightarrow A)\big]$	A
F	**T** T	F

2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$ — Invalid

A	B	$A \leftrightarrow \neg(B \leftrightarrow A)$	A
F	F	**T** F T	F

3. $A \rightarrow B, B \therefore A$ — Invalid

A	B	$A \rightarrow B$	B	A
F	T	T	T	F

4. $A \vee B, B \vee C, \neg B \therefore A \wedge C$ — Valid

A	B	C	$A \vee B$	$B \vee C$	$\neg B$	$A \wedge C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F	T	F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$ — Valid

A	B	C	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T	T	F	F
F	F	F			T

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $A \to \neg A$ — Contingent

A	A	$\to$	$\neg$	A
T	T	**F**	F	T
F	F	**T**	T	F

2. $A \to (A \wedge (A \vee B))$ — Tautology

A	B	A	$\to$	$(A$	$\wedge$	$(A$	$\vee$	$B))$
T	T	T	**T**	T	T	T	T	T
T	F	T	**T**	T	T	T	T	F
F	T	F	**T**	F	F	F	T	T
F	F	F	**T**	F	F	F	F	F

3. $(A \to B) \leftrightarrow (B \to A)$ — Contingent

A	B	$(A$	$\to$	$B)$	$\leftrightarrow$	$(B$	$\to$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**F**	F	T	T
F	T	F	T	T	**F**	T	F	F
F	F	F	T	F	**T**	F	T	F

4. $A \to \neg(A \wedge (A \vee B))$ — Contingent

A	B	A	$\to$	$\neg$	$(A$	$\wedge$	$(A$	$\vee$	$B))$
T	T	T	**F**	F	T	T	T	T	T
T	F	T	**F**	F	T	T	T	T	F
F	T	F	**T**	T	F	F	F	T	T
F	F	F	**T**	T	F	F	F	F	F

5. $\neg B \to [(\neg A \wedge A) \vee B]$ — Contingent

A	B	$\neg$	B	$\to$	$((\neg$	A	$\wedge$	$A)$	$\vee$	$B)$
T	T	F	T	**T**	F	T	F	T	T	T
T	F	T	F	**F**	F	T	F	T	F	F
F	T	F	T	**T**	T	F	F	F	T	T
F	F	T	F	**F**	T	F	F	F	F	F

6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ — Tautology

A	B	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	A	$\wedge$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $[(A \wedge B) \wedge C] \rightarrow B$ Tautology

A	B	C	$((A$	$\wedge$	$B)$	$\wedge$	$C)$	$\rightarrow$	B
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

8. $\neg\big[(C \vee A) \vee B\big]$ Contingent

A	B	C	$\neg$	$((C$	$\vee$	$A)$	$\vee$	$B)$
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

9. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$ Contradiction

A	B	C	$((A$	$\wedge$	$B)$	$\wedge$	$\neg$	$(A$	$\wedge$	$B))$	$\wedge$	C
T	T	T	T	T	T	F	F	T	T	T	**F**	T
T	T	F	T	T	T	F	F	T	T	T	**F**	F
T	F	T	T	F	F	F	T	T	F	F	**F**	T
T	F	F	T	F	F	F	T	T	F	F	**F**	F
F	T	T	F	F	T	F	T	F	F	T	**F**	T
F	T	F	F	F	T	F	T	F	F	T	**F**	F
F	F	T	F	F	F	F	T	F	F	F	**F**	T
F	F	F	F	F	F	F	T	F	F	F	**F**	F

10. $(A \wedge B)] \rightarrow [(A \wedge C) \vee (B \wedge D)]$ Contingent

A	B	C	D	$((A$	$\wedge$	$B))$	$\rightarrow$	$((A$	$\wedge$	$C)$	$\vee$	$(B$	$\wedge$	$D))$
T	T	T	T	T	T	T	**T**	T	T	T	T	T	T	T
T	T	F	F	T	T	T	**F**	T	F	F	F	T	F	F

E. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $\neg(A \vee A)$ Contradiction
2. $(A \to B) \vee (B \to A)$ Tautology
3. $[(A \to B) \to A] \to A$ Tautology
4. $\neg[(A \to B) \vee (B \to A)]$ Contradiction
5. $(A \wedge B) \vee (A \vee B)$ Contingent
6. $\neg(A \wedge B) \leftrightarrow A$ Contingent
7. $A \to (B \vee C)$ Contingent
8. $(A \wedge \neg A) \to (B \vee C)$ Tautology
9. $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$ Contingent
10. $\neg[(A \to B) \vee (C \to D)]$ Contingent

F. Determine whether each the following pairs of sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "not equivalent."

1. A and $A \vee A$
2. A and $A \wedge A$
3. $A \vee \neg B$ and $A \to B$
4. $(A \to B)$ and $(\neg B \to \neg A)$
5. $\neg(A \wedge B)$ and $\neg A \vee \neg B$
6. $((U \to (X \vee X)) \vee U)$ and $\neg(X \wedge (X \wedge U))$
7. $((C \wedge (N \leftrightarrow C)) \leftrightarrow C)$ and $(\neg\neg\neg N \to C)$
8. $[(A \vee B) \wedge C]$ and $[A \vee (B \wedge C)]$
9. $((L \wedge C) \wedge I)$ and $L \vee C$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent. Justify your answer with a complete or partial truth table where appropriate.

1. $A \to A$, $\neg A \to \neg A$, $A \wedge A$, $A \vee A$ Consistent
2. $A \to \neg A$, $\neg A \to A$ Inconsistent
3. $A \vee B$, $A \to C$, $B \to C$ Consistent
4. $A \vee B$, $A \to C$, $B \to C$, $\neg C$ Inconsistent
5. $B \wedge (C \vee A)$, $A \to B$, $\neg(B \vee C)$ Inconsistent
6. $(A \leftrightarrow B) \to B$, $B \to \neg(A \leftrightarrow B)$, $A \vee B$ Consistent
7. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$ Consistent
8. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \to B$ Consistent
9. $A \leftrightarrow B$, $A \to C$, $B \to D$, $\neg(C \vee D)$ Consistent
10. $\neg(A \wedge \neg B)$, $B \to \neg A$, $\neg B$ Consistent

H. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \to (A \wedge \neg A) \therefore \neg A$ Valid

2. $A \vee B$, $A \to B$, $B \to A$ $\therefore$ $A \leftrightarrow B$ — Valid
3. $A \vee (B \to A)$ $\therefore$ $\neg A \to \neg B$ — Valid
4. $A \vee B$, $A \to B$, $B \to A$ $\therefore$ $A \wedge B$ — Valid
5. $(B \wedge A) \to C$, $(C \wedge A) \to B$ $\therefore$ $(C \wedge B) \to A$ — Invalid
6. $\neg(\neg A \vee \neg B)$, $A \to \neg C$ $\therefore$ $A \to (B \to C)$ — Invalid
7. $A \wedge (B \to C)$, $\neg C \wedge (\neg B \to \neg A)$ $\therefore$ $C \wedge \neg C$ — Valid
8. $A \wedge B$, $\neg A \to \neg C$, $B \to \neg D$ $\therefore$ $A \vee B$ — Invalid
9. $A \to B$ $\therefore$ $(A \wedge B) \vee (\neg A \wedge \neg B)$ — Invalid
10. $\neg A \to B, \neg B \to C, \neg C \to A$ $\therefore$ $\neg A \to (\neg B \vee \neg C)$ — Invalid

I. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \leftrightarrow \neg(B \leftrightarrow A)$ $\therefore$ A — Invalid
2. $A \vee B$, $B \vee C$, $\neg A$ $\therefore$ $B \wedge C$ — Invalid
3. $A \to C$, $E \to (D \vee B)$, $B \to \neg D$ $\therefore$ $(A \vee C) \vee (B \to (E \wedge D))$ — Invalid
4. $A \vee B$, $C \to A$, $C \to B$ $\therefore$ $A \to (B \to C)$ — Invalid
5. $A \to B$, $\neg B \vee A$ $\therefore$ $A \leftrightarrow B$ — Valid

CHAPTER 15

Basic rules for TFL

A. The following two 'proofs' are *incorrect*. Explain the mistakes they make.

1	$\neg L \to (A \land L)$	
2	$\quad \neg L$	
3	$\quad A$	$\to$E 1, 2
4	$\quad L$	
5	$\quad \bot$	$\bot$I 4, 2
6	$\quad A$	$\bot$E 5
7	A	TND 2–3, 4–6

$\to$E on line 3 should yield '$A \land L$'. 'A' could then be obtained by $\land$E.
$\bot$I on line 5 illicitly refers to a line from a closed subproof (line 2).

1	$A \land (B \land C)$	
2	$(B \lor C) \to D$	
3	B	$\land$E 1
4	$B \lor C$	$\lor$I 3
5	D	$\to$E 4, 2

$\land$E on line 3 should yield '$B \land C$'. 'B' could then be obtained by $\land$E again. The citation for line 5 is the wrong way round: it should be '$\to$E 2, 4'.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into *bona fide* proofs. Additionally, write down the argument that corresponds to each proof.

1	$P \land S$	
2	$S \to R$	
3	P	$\land$E 1
4	S	$\land$E 1
5	R	$\to$E 2, 4
6	$R \lor E$	$\lor$I 5

Corresponding argument:
$P \land S, S \to R \therefore R \lor E$

1	$A \to D$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad D$	$\to$E 1, 3
5	$\quad D \vee E$	$\vee$I 4
6	$(A \wedge B) \to (D \vee E)$	$\to$I 2–5

Corresponding argument:
$A \to D \therefore (A \wedge B) \to (D \vee E)$

1	$\neg L \rightarrow (J \vee L)$	
2	$\neg L$	
3	$J \vee L$	$\rightarrow$E 1, 2
4	$\quad J$	
5	$\quad J \wedge J$	$\wedge$I 4, 4
6	$\quad J$	$\wedge$E 5
7	$\quad L$	
8	$\quad \bot$	$\bot$I 7, 2
9	$\quad J$	$\bot$E 8
10	J	$\vee$E 3, 4–6, 7–9

Corresponding argument:
$\neg L \rightarrow (J \vee L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1. $J \rightarrow \neg J \therefore \neg J$

1	$J \rightarrow \neg J$	
2	$\quad J$	
3	$\quad \neg J$	$\rightarrow$E 1, 2
4	$\quad \bot$	$\bot$I 2, 3
5	$\neg J$	$\neg$I 2–4

2. $Q \rightarrow (Q \wedge \neg Q) \therefore \neg Q$

1	$Q \rightarrow (Q \wedge \neg Q)$	
2	$\quad Q$	
3	$\quad Q \wedge \neg Q$	$\rightarrow$E 1, 2
4	$\quad \neg Q$	$\wedge$E 3
5	$\quad \bot$	$\bot$I 2, 4
6	$\neg Q$	$\neg$I 2–5

3. $A \rightarrow (B \rightarrow C) \therefore (A \wedge B) \rightarrow C$

1	$A \rightarrow (B \rightarrow C)$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad B \rightarrow C$	$\rightarrow$E 1, 3
5	$\quad B$	$\wedge$E 2
6	$\quad C$	$\rightarrow$E 4, 5
7	$(A \wedge B) \rightarrow C$	$\rightarrow$I 2–6

4. $K \wedge L \therefore K \leftrightarrow L$

1	$K \wedge L$	
2	$\quad K$	
3	$\quad L$	$\wedge$E 1
4	$\quad L$	
5	$\quad K$	$\wedge$E 1
6	$K \leftrightarrow L$	$\leftrightarrow$I 2–3, 4–5

5. $(C \wedge D) \vee E \therefore E \vee D$

1	$(C \wedge D) \vee E$	
2	$\quad C \wedge D$	
3	$\quad D$	$\wedge$E 2
4	$\quad E \vee D$	$\vee$I 3
5	$\quad E$	
6	$\quad E \vee D$	$\vee$I 5
7	$E \vee D$	$\vee$E 1, 2–4, 5–6

6. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3	$\quad A$	
4	$\quad B$	$\leftrightarrow$E 1, 3
5	$\quad C$	$\leftrightarrow$E 2, 4
6	$\quad C$	
7	$\quad B$	$\leftrightarrow$E 2, 6
8	$\quad A$	$\leftrightarrow$E 1, 7
9	$A \leftrightarrow C$	$\leftrightarrow$I 3–5, 6–8

7. $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$

1	$\neg F \to G$	
2	$F \to H$	
3	$\quad F$	
4	$\quad H$	$\to$E 2, 3
5	$\quad G \vee H$	$\vee$I 4
6	$\quad \neg F$	
7	$\quad G$	$\to$E 1, 6
8	$\quad G \vee H$	$\vee$I 7
9	$G \vee H$	TND 3–5, 6–8

8. $(Z \wedge K) \vee (K \wedge M), K \rightarrow D \therefore D$

1	$(Z \wedge K) \vee (K \wedge M)$	
2	$K \rightarrow D$	
3	$\quad Z \wedge K$	
4	$\quad K$	$\wedge$E 3
5	$\quad K \wedge M$	
6	$\quad K$	$\wedge$E 5
7	K	$\vee$E 1, 3–4, 5–6
8	D	$\rightarrow$E 2, 7

9. $P \wedge (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$P \rightarrow \neg R$	
3	P	$\wedge$E 1
4	$\neg R$	$\rightarrow$E 2, 3
5	$Q \vee R$	$\wedge$E 1
6	$\quad Q$	
7	$\quad Q \vee E$	$\vee$I 6
8	$\quad R$	
9	$\quad \bot$	$\bot$I 8, 4
10	$\quad Q \vee E$	$\bot$E 9
11	$Q \vee E$	$\vee$E 5, 6–7, 8–10

10. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	
2	$\mid\ S$	
3	$\mid\ T$	$\leftrightarrow$E 1, 2
4	$\mid\ T \lor S$	$\lor$I 3
5	$\mid\ T \lor S$	
6	$\mid\ \mid\ T$	
7	$\mid\ \mid\ S$	$\leftrightarrow$E 1, 6
8	$\mid\ \mid\ S$	
9	$\mid\ \mid\ S \land S$	$\land$I 8, 8
10	$\mid\ \mid\ S$	$\land$E 9
11	$\mid\ S$	$\lor$E 5, 6–7, 8–10
12	$S \leftrightarrow (T \lor S)$	$\leftrightarrow$I 2–4, 5–11

11. $\neg(P \to Q) \therefore \neg Q$

1	$\neg(P \to Q)$	
2	$\mid\ Q$	
3	$\mid\ \mid\ P$	
4	$\mid\ \mid\ Q \wedge Q$	$\wedge$I 2, 2
5	$\mid\ \mid\ Q$	$\wedge$E 4
6	$\mid\ P \to Q$	$\to$I 3–5
7	$\mid\ \bot$	$\bot$I 6, 1
8	$\neg Q$	$\neg$I 2–7

12. $\neg(P \to Q) \therefore P$

1	$\neg(P \to Q)$	
2	$\mid\ P$	
3	$\mid\ P \wedge P$	$\wedge$I 2, 2
4	$\mid\ P$	$\wedge$E 3
5	$\mid\ \neg P$	
6	$\mid\ \mid\ P$	
7	$\mid\ \mid\ \bot$	$\bot$I 6, 5
8	$\mid\ \mid\ Q$	$\bot$E 7
9	$\mid\ P \to Q$	$\to$I 6–8
10	$\mid\ \bot$	$\bot$I 9, 1
11	$\mid\ P$	$\bot$E 10
12	P	TND 2–4, 5–11

CHAPTER 16

Additional rules for TFL

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \to \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	W	$\wedge$E 2
5	$\neg B$	$\to$E 1, 4
6	$J \wedge K$	DS 3, 5
7	K	$\wedge$E 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\quad \neg L$	
4	$\quad \neg O$	DS 2, 3
5	$\quad L$	$\leftrightarrow$E 1, 4
6	$\quad \bot$	$\bot$I 5, 3
7	$\neg\neg L$	$\neg$I 3–6
8	L	DNE 7

1	$Z \to (C \wedge \neg N)$	
2	$\neg Z \to (N \wedge \neg C)$	
3	$\quad \neg(N \vee C)$	
4	$\quad \neg N \wedge \neg C$	DeM 3
5	$\quad \neg N$	$\wedge$E 4
6	$\quad \neg C$	$\wedge$E 4
7	$\quad\quad Z$	
8	$\quad\quad C \wedge \neg N$	$\to$E 1, 7
9	$\quad\quad C$	$\wedge$E 8
10	$\quad\quad \bot$	$\bot$I 9, 6
11	$\quad \neg Z$	$\neg$I 7–10
12	$\quad N \wedge \neg C$	$\to$E 2, 11
13	$\quad N$	$\wedge$E 12
14	$\quad \bot$	$\bot$I 13, 5
15	$\neg\neg(N \vee C)$	$\neg$I 3–14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

1. $E \vee F, F \vee G, \neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \wedge G$	$\wedge$I 4, 5

2. $M \vee (N \to M) \therefore \neg M \to \neg N$

1	$M \vee (N \to M)$	
2	$\quad \neg M$	
3	$\quad N \to M$	DS 1, 2
4	$\quad \neg N$	MT 3, 2
5	$\neg M \to \neg N$	→I 2–4

3. $(M \vee N) \wedge (O \vee P), N \to P, \neg P \therefore M \wedge O$

1	$(M \vee N) \wedge (O \vee P)$	
2	$N \to P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	∧E 1
6	M	DS 5, 4
7	$O \vee P$	∧E 1
8	O	DS 7, 3
9	$M \wedge O$	∧I 6, 8

4. $(X \wedge Y) \vee (X \wedge Z), \neg(X \wedge D), D \vee M \therefore M$

1	$(X \wedge Y) \vee (X \wedge Z)$	
2	$\neg(X \wedge D)$	
3	$D \vee M$	
4	$\quad X \wedge Y$	
5	$\quad X$	∧E 4
6	$\quad X \wedge Z$	
7	$\quad X$	∧E 6
8	X	∨E 1, 4–5, 6–7
9	$\quad D$	
10	$\quad X \wedge D$	∧I 8, 9
11	$\quad \bot$	⊥I 10, 2
12	$\neg D$	¬I 9–11
13	M	DS 3, 12

CHAPTER 17

Proof-theoretic concepts

A. Show that each of the following sentences is a theorem:

1. $O \to O$

1	$\quad\mid O$	
2	$\quad\mid O$	R 1
3	$O \to O$	$\to$I 1–2

2. $N \lor \neg N$

1	$\quad\mid N$	
2	$\quad\mid N \lor \neg N$	$\lor$I 1
3	$\quad\mid \neg N$	
4	$\quad\mid N \lor \neg N$	$\lor$I 3
5	$N \lor \neg N$	TND 1–2, 3–4

3. $J \leftrightarrow [J \lor (L \land \neg L)]$

1	J	
2	$J \vee (L \wedge \neg L)$	$\vee$I 1
3	$J \vee (L \wedge \neg L)$	
4	$L \wedge \neg L$	
5	L	$\wedge$E 4
6	$\neg L$	$\wedge$E 4
7	$\bot$	$\bot$I 5, 6
8	$\neg(L \wedge \neg L)$	$\neg$I 4–7
9	J	DS 3, 8
10	$J \leftrightarrow [J \vee (L \wedge \neg L)]$	$\leftrightarrow$I 1–2, 3–9

4. $((A \to B) \to A) \to A$

1	$(A \to B) \to A$	
2	$\neg A$	
3	$\neg(A \to B)$	MT 1, 2
4	A	
5	$\bot$	$\bot$I 4, 2
6	B	$\bot$E 5
7	$A \to B$	$\to$I 4–6
8	$\bot$	$\bot$I 7, 3
9	$\neg\neg A$	$\neg$I 2
10	A	DNE 9
11	$((A \to B) \to A) \to A$	$\to$I 1–10

B. Provide proofs to show each of the following:

1. $C \to (E \wedge G), \neg C \to G \vdash G$

1	$C \to (E \wedge G)$	
2	$\neg C \to G$	
3	C	
4	$E \wedge G$	$\to$E 1, 3
5	G	$\wedge$E 4
6	$\neg C$	
7	G	$\to$E 2, 6
8	G	TND 3–5, 6–7

2. $M \wedge (\neg N \to \neg M) \vdash (N \wedge M) \vee \neg M$

1	$M \wedge (\neg N \rightarrow \neg M)$	
2	M	$\wedge$E 1
3	$\neg N \rightarrow \neg M$	$\wedge$E 1
4	$\quad \neg N$	
5	$\quad \neg M$	$\rightarrow$E 3, 4
6	$\quad \bot$	$\bot$I 2, 5
7	$\neg\neg N$	$\neg$I 4–6
8	N	DNE 7
9	$N \wedge M$	$\wedge$I 8, 2
10	$(N \wedge M) \vee \neg M$	$\vee$I 9

3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \rightarrow M) \vdash Y \rightarrow Z$

1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$\underline{D \wedge (D \rightarrow M)}$	
3	D	$\wedge$E 2
4	$D \rightarrow M$	$\wedge$E 2
5	M	$\rightarrow$E 4, 3
6	$\quad \mid \underline{Y}$	
7	$\quad \mid Y \wedge M$	$\wedge$I 6, 5
8	$\quad \mid Z \wedge K$	$\leftrightarrow$E 1, 7
9	$\quad \mid Z$	$\wedge$E 8
10	$Y \rightarrow Z$	$\rightarrow$I 6–9

4. $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\underline{\neg Z}$	
4	$\quad \mid \underline{W \vee X}$	
5	$\quad \mid \quad \mid \underline{W}$	
6	$\quad \mid \quad \mid W \vee Y$	$\vee$I 5
7	$\quad \mid \quad \mid \underline{X}$	
8	$\quad \mid \quad \mid Y$	$\rightarrow$E 2, 7
9	$\quad \mid \quad \mid W \vee Y$	$\vee$I 8
10	$\quad \mid W \vee Y$	$\vee$E 4, 5–6, 7–9
11	$\quad \mid \underline{Y \vee Z}$	
12	$\quad \mid Y$	DS 11, 3
13	$\quad \mid W \vee Y$	$\vee$I 12
14	$W \vee Y$	$\vee$E 1, 4–10, 11–13

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E$, $E \leftrightarrow R$

1	$\underline{R \leftrightarrow E}$	
2	$\quad \mid \underline{E}$	
3	$\quad \mid R$	$\leftrightarrow$E 1, 2
4	$\quad \mid \underline{R}$	
5	$\quad \mid E$	$\leftrightarrow$E 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow$I 2–3, 4–5

1	$E \leftrightarrow R$	
2	$\quad E$	
3	$\quad R$	$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	$\leftrightarrow$E 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow$I 4–5, 2–3

2. G, $\neg\neg\neg\neg G$

1	G	
2	$\quad \neg\neg\neg G$	
3	$\quad \neg G$	DNE 2
4	$\quad \bot$	$\bot$I 1, 3
5	$\neg\neg\neg\neg G$	$\neg$I 2–4

1	$\neg\neg\neg\neg G$	
2	$\neg\neg G$	DNE 1
3	G	DNE 2

3. $T \to S$, $\neg S \to \neg T$

1	$T \to S$	
2	$\quad \neg S$	
3	$\quad \neg T$	MT 1, 2
4	$\neg S \to \neg T$	$\to$I 2–3

1	$\neg S \to \neg T$	
2	$\quad T$	
3	$\quad\quad \neg S$	
4	$\quad\quad \neg T$	$\to$E 1, 3
5	$\quad\quad \bot$	$\bot$I 2, 4
6	$\quad \neg\neg S$	$\neg$I 3–5
7	$\quad S$	DNE 6
8	$T \to S$	$\to$I 2–7

4. $U \to I$, $\neg(U \wedge \neg I)$

1	$U \to I$	
2	$\quad U \wedge \neg I$	
3	$\quad U$	$\wedge$E 2
4	$\quad \neg I$	$\wedge$E 2
5	$\quad I$	$\to$E 1, 3
6	$\quad \bot$	$\bot$I 5, 4
7	$\neg(U \wedge \neg I)$	$\neg$I 2–6

1	$\neg(U \wedge \neg I)$	
2	$\quad U$	
3	$\quad\quad \neg I$	
4	$\quad\quad U \wedge \neg I$	$\wedge$I 2, 3
5	$\quad\quad \bot$	$\bot$I 4, 1
6	$\quad \neg\neg I$	$\neg$I 3–5
7	$\quad I$	DNE 6
8	$U \to I$	$\to$I 2–7

5. $\neg(C \to D), C \land \neg D$

1	$\mid C \land \neg D$	
2	$\mid C$	$\land$E 1
3	$\mid \neg D$	$\land$E 1
4	$\mid \quad \mid C \to D$	
5	$\mid \quad \mid D$	$\to$E 4, 2
6	$\mid \quad \mid \bot$	$\bot$I 5, 3
7	$\mid \neg(C \to D)$	$\neg$I 4–6

1	$\mid \neg(C \to D)$	
2	$\mid \quad \mid D$	
3	$\mid \quad \mid \quad \mid C$	
4	$\mid \quad \mid \quad \mid D$	R 2
5	$\mid \quad \mid C \to D$	$\to$I 3–4
6	$\mid \quad \mid \bot$	$\bot$I 5, 1
7	$\mid \neg D$	$\neg$I 2–6
8	$\mid \quad \mid \neg C$	
9	$\mid \quad \mid \quad \mid C$	
10	$\mid \quad \mid \quad \mid \bot$	$\bot$I 9, 8
11	$\mid \quad \mid \quad \mid D$	$\bot$E 10
12	$\mid \quad \mid C \to D$	$\to$I 9–11
13	$\mid \quad \mid \bot$	$\bot$I 12, 1
14	$\mid \neg\neg C$	$\neg$I 8–13
15	$\mid C$	DNE 14
16	$\mid C \land \neg D$	$\land$I 15, 7

6. $\neg G \leftrightarrow H, \neg(G \leftrightarrow H)$

1	$\mid \neg G \leftrightarrow H$	
2	$\mid \quad \mid G \leftrightarrow H$	
3	$\mid \quad \mid \quad \mid G$	
4	$\mid \quad \mid \quad \mid H$	$\leftrightarrow$E 2, 3
5	$\mid \quad \mid \quad \mid \neg G$	$\leftrightarrow$E 1, 4
6	$\mid \quad \mid \quad \mid \bot$	$\bot$I 3, 5
7	$\mid \quad \mid \quad \mid \neg G$	
8	$\mid \quad \mid \quad \mid H$	$\leftrightarrow$E 1, 7
9	$\mid \quad \mid \quad \mid G$	$\leftrightarrow$E 2, 8
10	$\mid \quad \mid \quad \mid \bot$	$\bot$I 9, 7
11	$\mid \quad \mid \bot$	TND 3–6, 7–10
12	$\mid \neg(G \leftrightarrow H)$	$\neg$I 2–11

1	$\neg(G \leftrightarrow H)$	
2	$\quad \neg G$	
3	$\quad\quad \neg H$	
4	$\quad\quad\quad G$	
5	$\quad\quad\quad \bot$	$\bot$I 4, 2
6	$\quad\quad\quad H$	$\bot$E 5
7	$\quad\quad\quad H$	
8	$\quad\quad\quad \bot$	$\bot$I 7, 3
9	$\quad\quad\quad G$	$\bot$E 8
10	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 4–6, 7–9
11	$\quad\quad \bot$	$\bot$I 10, 1
12	$\quad \neg\neg H$	$\neg$I 3–11
13	$\quad H$	DNE 12
14	$\quad H$	
15	$\quad\quad G$	
16	$\quad\quad\quad G$	
17	$\quad\quad\quad H$	R 14
18	$\quad\quad\quad H$	
19	$\quad\quad\quad G$	R 15
20	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 16–17, 18–19
21	$\quad\quad \bot$	$\bot$I 20, 1
22	$\quad \neg G$	$\neg$I 15–21
23	$\neg G \leftrightarrow H$	$\leftrightarrow$I 2–13, 14–22

D. If you know that $\mathscr{A} \vdash \mathscr{B}$, what can you say about $(\mathscr{A} \wedge \mathscr{C}) \vdash \mathscr{B}$? What about $(\mathscr{A} \vee \mathscr{C}) \vdash \mathscr{B}$? Explain your answers.

If $\mathscr{A} \vdash \mathscr{B}$, then $(\mathscr{A} \wedge \mathscr{C}) \vdash \mathscr{B}$. After all, if $\mathscr{A} \vdash \mathscr{B}$, then there is some proof with assumption $\mathscr{A}$ that ends with $\mathscr{B}$, and no undischarged assumptions other than $\mathscr{A}$. Now, if we start a proof with assumption $(\mathscr{A} \wedge \mathscr{C})$, we can obtain $\mathscr{A}$ by $\wedge$E. We can now copy and paste the original proof of $\mathscr{B}$ from $\mathscr{A}$, adding 1 to every line number and line number citation. The result will be a proof of $\mathscr{B}$ from assumption $\mathscr{A}$.

However, we cannot prove much from $(\mathscr{A} \vee \mathscr{C})$. After all, it might be impossible to prove $\mathscr{B}$ from $\mathscr{C}$.

E. In this chapter, we claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem. Why did we claim this? (*Hint*: think of a sentence that would be a theorem iff $\mathcal{A}$ and $\mathcal{B}$ were provably equivalent.)

Consider the sentence $\mathcal{A} \leftrightarrow \mathcal{B}$. Suppose we can show that this is a theorem. So we can prove it, with no assumptions, in m lines, say. Then if we assume $\mathcal{A}$ and copy and paste the proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ (changing the line numbering), we will have a deduction of this shape:

1	$\mathcal{A}$	
$m+1$	$\mathcal{A} \leftrightarrow \mathcal{B}$	
$m+2$	$\mathcal{B}$	$\leftrightarrow$E $m+1$, 1

This will show that $\mathcal{A} \vdash \mathcal{B}$. In exactly the same way, we can show that $\mathcal{B} \vdash \mathcal{A}$. So if we can show that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem, we can show that $\mathcal{A}$ and $\mathcal{B}$ are provably equivalent.

Conversely, suppose we can show that $\mathcal{A}$ and $\mathcal{B}$ are provably equivalent. Then we can prove $\mathcal{B}$ from the assumption of $\mathcal{A}$ in m lines, say, and prove $\mathcal{A}$ from the assumption of $\mathcal{B}$ in n lines, say. Copying and pasting these proofs together (changing the line numbering where appropriate), we obtain:

1		$\mathcal{A}$	
m		$\mathcal{B}$	
$m+1$		$\mathcal{B}$	
$m+n$		$\mathcal{A}$	
$m+n+1$	$\mathcal{A} \leftrightarrow \mathcal{B}$		$\leftrightarrow$I 1–m, $m+1$–$m+n$

Thus showing that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem.

There was nothing special about $\mathcal{A}$ and $\mathcal{B}$ in this. So what this shows is that the problem of showing that two sentences are provably equivalent is, essentially, the same problem as showing that a certain kind of sentence (a biconditional) is a theorem.

CHAPTER 19
Derived rules

A. Provide proof schemes that justify the addition of the third and fourth De Morgan rules as derived rules.

Third rule:

m	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
k	$\neg\mathscr{A}$	$\wedge$E m
$k+1$	$\neg\mathscr{B}$	$\wedge$E m
$k+2$	$\quad \mathscr{A} \vee \mathscr{B}$	
$k+3$	$\quad\quad \mathscr{A}$	
$k+4$	$\quad\quad \bot$	$\bot$I $k+3$, k
$k+5$	$\quad\quad \mathscr{B}$	
$k+6$	$\quad\quad \bot$	$\bot$I $k+5$, $k+1$
$k+7$	$\quad \bot$	$\vee$E $k+2$, $k+3$–$k+4$, $k+5$–$k+6$
$k+8$	$\neg(\mathscr{A} \vee \mathscr{B})$	$\neg$I $k+2$–$k+7$

Fourth rule:

m	$\neg(\mathscr{A} \vee \mathscr{B})$	
k	$\quad \mathscr{A}$	
$k+1$	$\quad \mathscr{A} \vee \mathscr{B}$	$\vee$I k
$k+2$	$\quad \bot$	$\bot$I $k+1$, m
$k+3$	$\neg\mathscr{A}$	$\neg$I k–$k+2$
$k+4$	$\quad \mathscr{B}$	
$k+5$	$\quad \mathscr{A} \vee \mathscr{B}$	$\vee$I $k+4$
$k+6$	$\quad \bot$	$\bot$I $k+5$, m
$k+7$	$\neg\mathscr{B}$	$\neg$I $k+4$–$k+6$
$k+8$	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	$\wedge$I $k+3$, $k+7$

B. The proofs you offered in response to the practice exercises of §§16–17 used derived rules. Replace the use of derived rules, in such proofs, with only basic rules. You will find some 'repetition' in the resulting proofs; in such cases, offer a streamlined proof using only basic rules. (This will give you a sense, both of the power of derived rules, and of how all the rules interact.)

CHAPTER 20

Soundness and Completeness

Practice exercises

A. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow [((B \wedge C) \vee D) \rightarrow A]$ is a tautology.
2. Show that $A \rightarrow (A \rightarrow B)$ is not a tautology
3. Show that the sentence $A \rightarrow \neg A$ is not a contradiction.
4. Show that the sentence $A \leftrightarrow \neg A$ is a contradiction.
5. Show that the sentence $\neg(W \rightarrow (J \vee J))$ is contingent
6. Show that the sentence $\neg(X \vee (Y \vee Z)) \vee (X \vee (Y \vee Z))$ is not contingent
7. Show that the sentence $B \rightarrow \neg S$ is equivalent to the sentence $\neg\neg B \rightarrow \neg S$
8. Show that the sentence $\neg(X \vee O)$ is not equivalent to the sentence $X \wedge O$
9. Show that the sentences $\neg(A \vee B)$, C, $C \rightarrow A$ are jointly inconsistent.
10. Show that the sentences $\neg(A \vee B)$, $\neg B$, $B \rightarrow A$ are jointly consistent
11. Show that $\neg(A \vee (B \vee C)) \therefore \neg C$ is valid.
12. Show that $\neg(A \wedge (B \vee C)) \therefore \neg C$ is invalid.

B. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow (B \rightarrow A)$ is a tautology
2. Show that $\neg(((N \leftrightarrow Q) \vee Q) \vee N)$ is not a tautology

3. Show that $Z \vee (\neg Z \leftrightarrow Z)$ is contingent
4. show that $(L \leftrightarrow ((N \rightarrow N) \rightarrow L)) \vee H$ is not contingent
5. Show that $(A \leftrightarrow A) \wedge (B \wedge \neg B)$ is a contradiction
6. Show that $(B \leftrightarrow (C \vee B))$ is not a contradiction.
7. Show that $((\neg X \leftrightarrow X) \vee X)$ is equivalent to X
8. Show that $F \wedge (K \wedge R)$ is not equivalent to $(F \leftrightarrow (K \leftrightarrow R))$
9. Show that the sentences $\neg(W \rightarrow W)$, $(W \leftrightarrow W) \wedge W$, $E \vee (W \rightarrow \neg(E \wedge W))$ are inconsistent.
10. Show that the sentences $\neg R \vee C$, $(C \wedge R) \rightarrow \neg R$, $(\neg(R \vee R) \rightarrow R)$ are consistent.
11. Show that $\neg\neg(C \leftrightarrow \neg C), ((G \vee C) \vee G) \therefore ((G \rightarrow C) \wedge G)$ is valid.
12. Show that $\neg\neg L, (C \rightarrow \neg L) \rightarrow C) \therefore \neg C$ is invalid.

CHAPTER 22

Sentences with one quantifier

A. Here are the syllogistic figures identified by Aristotle and his successors, along with their medieval names:

- **Barbara**. All G are F. All H are G. So: All H are F
  $\forall x(Gx \to Fx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to Fx)$
- **Celarent**. No G are F. All H are G. So: No H are F
  $\forall x(Gx \to \neg Fx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Ferio**. No G are F. Some H is G. So: Some H is not F
  $\forall x(Gx \to \neg Fx), \exists x(Hx \land Gx) \therefore \exists x(Hx \land \neg Fx)$
- **Darii**. All G are H. Some H is G. So: Some H is F.
  $\forall x(Gx \to Fx), \exists x(Hx \land Gx) \therefore \exists x(Hx \land Fx)$
- **Camestres**. All F are G. No H are G. So: No H are F.
  $\forall x(Fx \to Gx), \forall x(Hx \to \neg Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Cesare**. No F are G. All H are G. So: No H are F.
  $\forall x(Fx \to \neg Gx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Baroko**. All F are G. Some H is not G. So: Some H is not F.
  $\forall x(Fx \to Gx), \exists x(Hx \land \neg Gx) \therefore \exists x(Hx \land \neg Fx)$
- **Festino**. No F are G. Some H are G. So: Some H is not F.
  $\forall x(Fx \to \neg Gx), \exists x(Hx \land Gx) \therefore \exists x(Hx \land \neg Fx)$
- **Datisi**. All G are F. Some G is H. So: Some H is F.
  $\forall x(Gx \to Fx), \exists x(Gx \land Hx) \therefore \exists x(Hx \land Fx)$
- **Disamis**. Some G is F. All G are H. So: Some H is F.
  $\exists x(Gx \land Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$
- **Ferison**. No G are F. Some G is H. So: Some H is not F.
  $\forall x(Gx \to \neg Fx), \exists x(Gx \land Hx) \therefore \exists x(Hx \land \neg Fx)$
- **Bokardo**. Some G is not F. All G are H. So: Some H is not F.
  $\exists x(Gx \land \neg Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land \neg Fx)$
- **Camenes**. All F are G. No G are H So: No H is F.
  $\forall x(Fx \to Gx), \forall x(Gx \to \neg Hx) \therefore \forall x(Hx \to \neg Fx)$

- **Dimaris.** Some F is G. All G are H. So: Some H is F.
  $\exists x(Fx \land Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$
- **Fresison.** No F are G. Some G is H. So: Some H is not F.
  $\forall x(Fx \to \neg Gx), \exists x(Gx \land Hx) \therefore \exists(Hx \land \neg Fx)$

Symbolize each argument in FOL.

B. Using the following symbolization key:

domain: people
Kx: $\underline{\quad}_x$ knows the combination to the safe
Sx: $\underline{\quad}_x$ is a spy
Vx: $\underline{\quad}_x$ is a vegetarian
h: Hofthor
i: Ingmar

symbolize the following sentences in FOL:

1. Neither Hofthor nor Ingmar is a vegetarian.
   $\neg Vh \land \neg Vi$
2. No spy knows the combination to the safe.
   $\forall x(Sx \to \neg Kx)$
3. No one knows the combination to the safe unless Ingmar does.
   $\forall x \neg Kx \lor Ki$
4. Hofthor is a spy, but no vegetarian is a spy.
   $Sh \land \forall x(Vx \to \neg Sx)$

C. Using this symbolization key:

domain: all animals
Ax: $\underline{\quad}_x$ is an alligator.
Mx: $\underline{\quad}_x$ is a monkey.
Rx: $\underline{\quad}_x$ is a reptile.
Zx: $\underline{\quad}_x$ lives at the zoo.
a: Amos
b: Bouncer
c: Cleo

symbolize each of the following sentences in FOL:

1. Amos, Bouncer, and Cleo all live at the zoo.
   $Za \land Zb \land Zc$
2. Bouncer is a reptile, but not an alligator.
   $Rb \land \neg Ab$
3. Some reptile lives at the zoo.
   $\exists x(Rx \land Zx)$
4. Every alligator is a reptile.
   $\forall x(Ax \to Rx)$
5. Any animal that lives at the zoo is either a monkey or an alligator.

$\forall x(Zx \to (Mx \lor Ax))$

6. There are reptiles which are not alligators.
   $\exists x(Rx \land \neg Ax)$
7. If any animal is an reptile, then Amos is.
   $\exists xRx \to Ra$
8. If any animal is an alligator, then it is a reptile.
   $\forall x(Ax \to Rx)$

D. For each argument, write a symbolization key and symbolize the argument in FOL.

1. Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat

   domain: people
   Lx: $\_\_\_\_\__x$ is a logician
   Hx: $\_\_\_\_\__x$ wears a funny hat
   i: Willard

   $Li, \forall x(Lx \to Hx) \therefore Hi$
2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.

   domain: physical things
   Dx: $\_\_\_\_\__x$ is on my desk
   Ex: $\_\_\_\_\__x$ escapes my attention
   Cx: $\_\_\_\_\__x$ is a computer

   $\forall x(Dx \to \neg Ex), \exists x(Dx \land Cx) \therefore \exists x(Cx \land \neg Ex)$
3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.

   domain: episodes (psychological and televised)
   Dx: $\_\_\_\_\__x$ is one of my dreams
   Bx: $\_\_\_\_\__x$ is in black and white
   Ox: $\_\_\_\_\__x$ is an old TV show

   $\forall x(Dx \to Bx), \forall x(Ox \to Bx) \therefore \exists x(Dx \land Ox)$.
   Comment: generic statements are tricky to deal with. Does the second sentence mean that *all* old TV shows are in black and white; or that most of them are; or that most of the things which are in black and white are old TV shows? I have gone with the former, but it is not clear that FOL deals with these well.
4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.

   domain: people
   Ax: $\_\_\_\_\__x$ has been to Australia
   Kx: $\_\_\_\_\__x$ has seen a kangaroo

Zx: $\_\_\_\_\__x$ has been to a zoo
h: Holmes
a: Watson

$\neg Ah \land \neg Aa, \forall x(Kx \rightarrow (Ax \lor Zx)), \neg Ka \land Kh \therefore Zh$

5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

domain: people
Sx: $\_\_\_\_\__x$ expects the Spanish Inquisition
Tx: $\_\_\_\_\__x$ knows the troubles I've seen
h: Holmes
a: Watson

$\forall x \neg Sx, \forall x \neg Tx \therefore \forall x(Sx \rightarrow Tx)$

6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

domain: people
Bx: $\_\_\_\_\__x$ is a baby
Ix: $\_\_\_\_\__x$ is illogical
Cx: $\_\_\_\_\__x$ can manage a crocodile
b: Berthold

$\forall x(Bx \rightarrow Ix), \forall x(Ix \rightarrow \neg Cx), Bb \therefore \neg Cb$

CHAPTER 23

Multiple generality

A. Using this symbolization key:

domain: all animals
Ax: $\underline{\hspace{2em}}_x$ is an alligator
Mx: $\underline{\hspace{2em}}_x$ is a monkey
Rx: $\underline{\hspace{2em}}_x$ is a reptile
Zx: $\underline{\hspace{2em}}_x$ lives at the zoo
Lxy: $\underline{\hspace{2em}}_x$ loves $\underline{\hspace{2em}}_y$
a: Amos
b: Bouncer
c: Cleo

symbolize each of the following sentences in FOL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.
   $Lcb \rightarrow Mb$
2. If both Bouncer and Cleo are alligators, then Amos loves them both.
   $(Ab \wedge Ac) \rightarrow (Lab \wedge Lac)$
3. Cleo loves a reptile.
   $\exists x(Rx \wedge Lcx)$
   Comment: this English expression is ambiguous; in some contexts, it can be read as a generic, along the lines of 'Cleo loves reptiles'. (Compare 'I do love a good pint'.)
4. Bouncer loves all the monkeys that live at the zoo.
   $\forall x((Mx \wedge Zx) \rightarrow Lbx)$
5. All the monkeys that Amos loves love him back.
   $\forall x((Mx \wedge Lax) \rightarrow Lxa)$
6. Every monkey that Cleo loves is also loved by Amos.
   $\forall x((Mx \wedge Lcx) \rightarrow Lax)$

7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.
   $\exists x(Mx \wedge Lxb \wedge \neg Lbx)$

B. Using the following symbolization key:

domain: all animals
Dx: _____$_x$ is a dog
Sx: _____$_x$ likes samurai movies
Lxy: _____$_x$ is larger than _____$_y$
r: Rave
h: Shane
d: Daisy

symbolize the following sentences in FOL:

1. Rave is a dog who likes samurai movies.
   $Dr \wedge Sr$
2. Rave, Shane, and Daisy are all dogs.
   $Dr \wedge Dh \wedge Dd$
3. Shane is larger than Rave, and Daisy is larger than Shane.
   $Lhr \wedge Ldh$
4. All dogs like samurai movies.
   $\forall x(Dx \rightarrow Sx)$
5. Only dogs like samurai movies.
   $\forall x(Sx \rightarrow Dx)$
   Comment: the FOL sentence just written does not require that anyone likes samurai movies. The English sentence might suggest that at least some dogs *do* like samurai movies?
6. There is a dog that is larger than Shane.
   $\exists x(Dx \wedge Lxh)$
7. If there is a dog larger than Daisy, then there is a dog larger than Shane.
   $\exists x(Dx \wedge Lxd) \rightarrow \exists x(Dx \wedge Lxh)$
8. No animal that likes samurai movies is larger than Shane.
   $\forall x(Sx \rightarrow \neg Lxh)$
9. No dog is larger than Daisy.
   $\forall x(Dx \rightarrow \neg Lxd)$
10. Any animal that dislikes samurai movies is larger than Rave.
    $\forall x(\neg Sx \rightarrow Lxr)$
    Comment: this is very poor, though! For 'dislikes' does not mean the same as 'does not like'.
11. There is an animal that is between Rave and Shane in size.
    $\exists x((Lbx \wedge Lxh) \vee (Lhx \wedge Lxr))$
12. There is no dog that is between Rave and Shane in size.
    $\forall x\big(Dx \rightarrow \neg\big[(Lbx \wedge Lxh) \vee (Lhx \wedge Lxr)\big]\big)$
13. No dog is larger than itself.
    $\forall x(Dx \rightarrow \neg Lxx)$

14. Every dog is larger than some dog.
    $\forall x(Dx \rightarrow \exists y(Dy \land Lxy))$
    Comment: the English sentence is potentially ambiguous here. I have resolved the ambiguity by assuming it should be paraphrased by 'for every dog, there is a dog smaller than it'.
15. There is an animal that is smaller than every dog.
    $\exists x \forall y(Dy \rightarrow Lyx)$
16. If there is an animal that is larger than any dog, then that animal does not like samurai movies.
    $\forall x(\forall y(Dy \rightarrow Lxy) \rightarrow \neg Sx)$
    Comment: I have assumed that 'larger than any dog' here means 'larger than every dog'.

C. Using the symbolization key given, translate each English-language sentence into FOL.

domain: candies
Cx: _____$_x$ has chocolate in it.
Mx: _____$_x$ has marzipan in it.
Sx: _____$_x$ has sugar in it.
Tx: Boris has tried _____$_x$.
Bxy: _____$_x$ is better than _____$_y$.

1. Boris has never tried any candy.
2. Marzipan is always made with sugar.
3. Some candy is sugar-free.
4. The very best candy is chocolate.
5. No candy is better than itself.
6. Boris has never tried sugar-free chocolate.
7. Boris has tried marzipan and chocolate, but never together.
8. Any candy with chocolate is better than any candy without it.
9. Any candy with chocolate and marzipan is better than any candy that lacks both.

D. Using the following symbolization key:

domain: people and dishes at a potluck
Rx: _____$_x$ has run out.
Tx: _____$_x$ is on the table.
Fx: _____$_x$ is food.
Px: _____$_x$ is a person.
Lxy: _____$_x$ likes _____$_y$.
e: Eli
f: Francesca
g: the guacamole

symbolize the following English sentences in FOL:

1. All the food is on the table.
   $\forall x(Fx \to Tx)$
2. If the guacamole has not run out, then it is on the table.
   $\neg Rg \to Tg$
3. Everyone likes the guacamole.
   $\forall x(Px \to Lxg)$
4. If anyone likes the guacamole, then Eli does.
   $\exists x(Px \land Lxg) \to Leg$
5. Francesca only likes the dishes that have run out.
   $\forall x\big[(Lfx \land Fx) \to Rx\big]$
6. Francesca likes no one, and no one likes Francesca.
   $\forall x\big[Px \to (\neg Lfx \land \neg Lxf)\big]$
7. Eli likes anyone who likes the guacamole.
   $\forall x((Px \land Lxg) \to Lex)$
8. Eli likes anyone who likes the people that he likes.
   $\forall x\big[\big(Px \land \forall y[(Py \land Ley) \to Lxy]\big) \to Lex\big]$
9. If there is a person on the table already, then all of the food must have run out.
   $\exists x(Px \land Tx) \to \forall x(Fx \to Rx)$

E. Using the following symbolization key:

domain: people
Dx: _____$_x$ dances ballet.
Fx: _____$_x$ is female.
Mx: _____$_x$ is male.
Cxy: _____$_x$ is a child of _____$_y$.
Sxy: _____$_x$ is a sibling of _____$_y$.
e: Elmer
j: Jane
p: Patrick

symbolize the following sentences in FOL:

1. All of Patrick's children are ballet dancers.
   $\forall x(Cxp \to Dx)$
2. Jane is Patrick's daughter.
   $Cjp \land Fj$
3. Patrick has a daughter.
   $\exists x(Cxp \land Fx)$
4. Jane is an only child.
   $\neg\exists x Sxj$
5. All of Patrick's sons dance ballet.
   $\forall x\big[(Cxp \land Mx) \to Dx\big]$
6. Patrick has no sons.
   $\neg\exists x(Cxp \land Mx)$
7. Jane is Elmer's niece.

$\exists x(Sxe \land Cjx \land Fj)$

8. Patrick is Elmer's brother.
   $Spe \land Mp$
9. Patrick's brothers have no children.
   $\forall x\big[(Spx \land Mx) \rightarrow \neg\exists y Cyx\big]$
10. Jane is an aunt.
    $Fj \land \exists x(Sxj \land \exists y Cyx)$
11. Everyone who dances ballet has a brother who also dances ballet.
    $\forall x\big[Dx \rightarrow \exists y(My \land Syx \land Dy)\big]$
12. Every woman who dances ballet is the child of someone who dances ballet.
    $\forall x\big[(Fx \land Dx) \rightarrow \exists y(Cxy \land Dy)\big]$

CHAPTER 24

Identity

A. Explain why:

- '$\exists x \forall y(Ay \leftrightarrow x = y)$' is a good symbolization of 'there is exactly one apple'.
  We might naturally read this in English thus:
  - There is something, x, such that, if you choose any object at all, if you chose an apple then you chose x itself, and if you chose x itself then you chose an apple.

  The x in question must therefore be the one and only thing which is an apple.
- '$\exists x \exists y\big[\neg x = y \wedge \forall z(Az \leftrightarrow (x = z \vee y = z)\big]$' is a good symbolization of 'there are exactly two apples'.
  Similarly to the above, we might naturally read this in English thus:
  - There are two distinct things, x and y, such that if you choose any object at all, if you chose an apple then you either chose x or y, and if you chose either x or y then you chose an apple.

  The x and y in question must therefore be the only things which are apples, and since they are distinct, there are two of them.

CHAPTER 25

Definite descriptions

A. Using the following symbolization key:

domain: people
Kx: _____$_x$ knows the combination to the safe.
Sx: _____$_x$ is a spy.
Vx: _____$_x$ is a vegetarian.
Txy: _____$_x$ trusts _____$_y$.
h: Hofthor
i: Ingmar

symbolize the following sentences in FOL:

1. Hofthor trusts a vegetarian.
   $\exists x(Vx \land Thx)$
2. Everyone who trusts Ingmar trusts a vegetarian.
   $\forall x\big[Txi \to \exists y(Txy \land Vy)\big]$
3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.
   $\forall x\big[Txi \to \exists y\big(Txy \land \exists z(Tyz \land Vz)\big)\big]$
4. Only Ingmar knows the combination to the safe.
   $\forall x(Ki \to x = i)$
   Comment: does the English claim entail that Ingmar *does* know the combination to the safe? If so, then we should formalise this with a '$\leftrightarrow$'.
5. Ingmar trusts Hofthor, but no one else.
   $\forall x(Tix \leftrightarrow x = h)$
6. The person who knows the combination to the safe is a vegetarian.
   $\exists x\big[Kx \land \forall y(Ky \to x = y) \land Vx\big]$
7. The person who knows the combination to the safe is not a spy.
   $\exists x\big[Kx \land \forall y(Ky \to x = y) \land \neg Sx\big]$
   Comment: the scope of negation is potentially ambiguous here; I have read it as *inner* negation.

B. Using the following symbolization key:

domain: cards in a standard deck
Bx: $\_\_\_\_\__x$ is black.
Cx: $\_\_\_\_\__x$ is a club.
Dx: $\_\_\_\_\__x$ is a deuce.
Jx: $\_\_\_\_\__x$ is a jack.
Mx: $\_\_\_\_\__x$ is a man with an axe.
Ox: $\_\_\_\_\__x$ is one-eyed.
Wx: $\_\_\_\_\__x$ is wild.

symbolize each sentence in FOL:

1. All clubs are black cards.
   $\forall x(Cx \rightarrow Bx)$
2. There are no wild cards.
   $\neg\exists x Wx$
3. There are at least two clubs.
   $\exists x\exists y(\neg x = y \wedge Cx \wedge Cy)$
4. There is more than one one-eyed jack.
   $\exists x\exists y(\neg x = y \wedge Jx \wedge Ox \wedge Jy \wedge Oy)$
5. There are at most two one-eyed jacks.
   $\forall x\forall y\forall z\big[(Jx \wedge Ox \wedge Jy \wedge Oy \wedge Jz \wedge Oz) \rightarrow (x = y \vee x = z \vee y = z)\big]$
6. There are two black jacks.
   $\exists x\exists y(\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy)$
   Comment: I am reading this as 'there are *at least* two...'. If the suggestion was that there are *exactly* two, then a different FOL sentence would be required, namely:
   $\exists x\exists y\big(\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy \wedge \forall z[(Bz \wedge Jz) \rightarrow (x = z \vee y = z)]\big)$
7. There are four deuces.
   $\exists w\exists x\exists y\exists z(\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz)$
   Comment: I am reading this as 'there are *at least* four...'. If the suggestion is that there are *exactly* four, then we should offer instead:
   $\exists w\exists x\exists y\exists z\big(\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz \wedge \forall v[Dv \rightarrow (v = w \vee v = x \vee v = y \vee v = z)]\big)$
8. The deuce of clubs is a black card.
   $\exists x\big[Dx \wedge Cx \wedge \forall y\big((Dy \wedge Cy) \rightarrow x = y\big) \wedge Bx\big]$
9. One-eyed jacks and the man with the axe are wild.
   $\forall x\big[(Jx \wedge Ox) \rightarrow Wx\big] \wedge \exists x\big[Mx \wedge \forall y(My \rightarrow x = y) \wedge Wx\big]$
10. If the deuce of clubs is wild, then there is exactly one wild card.
    $\exists x\big(Dx \wedge Cx \wedge \forall y\big[(Dy \wedge Cy) \rightarrow x = y\big] \wedge Wx\big) \rightarrow \exists x\big(Wx \wedge \forall y(Wy \rightarrow x = y)\big)$
    Comment: if there is not exactly one deuce of clubs, then the above sentence is true. Maybe that's the wrong verdict. Perhaps the sentence should definitely be taken to imply that there is one and only one deuce of clubs, and then express a conditional about wildness. If so, then we might sym-

bolize it thus:
$\exists x\big(Dx \wedge Cx \wedge \forall y\big[(Dy \wedge Cy) \rightarrow x = y\big] \wedge \big[Wx \rightarrow \forall y(Wy \rightarrow x = y)\big]\big)$

11. The man with the axe is not a jack.
$\exists x\big[Mx \wedge \forall y(My \rightarrow x = y) \wedge \neg Jx\big]$
12. The deuce of clubs is not the man with the axe.
$\exists x \exists y\big(Dx \wedge Cx \wedge \forall z[(Dz \wedge Cz) \rightarrow x = z] \wedge My \wedge \forall z(Mz \rightarrow y = x) \wedge \neg x = y\big)$

C. Using the following symbolization key:

domain: animals in the world
Bx: $\underline{\quad\quad}_x$ is in Farmer Brown's field.
Hx: $\underline{\quad\quad}_x$ is a horse.
Px: $\underline{\quad\quad}_x$ is a Pegasus.
Wx: $\underline{\quad\quad}_x$ has wings.

symbolize the following sentences in FOL:

1. There are at least three horses in the world.
$\exists x \exists y \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz)$
2. There are at least three animals in the world.
$\exists x \exists y \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z)$
3. There is more than one horse in Farmer Brown's field.
$\exists x \exists y(\neg x = y \wedge Hx \wedge Hy \wedge Bx \wedge By)$
4. There are three horses in Farmer Brown's field.
$\exists x \exists y \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz \wedge Bx \wedge By \wedge Bz)$
Comment: I have read this as 'there are *at least* three…'. If the suggestion was that there are *exactly* three, then a different FOL sentence would be required.
5. There is a single winged creature in Farmer Brown's field; any other creatures in the field must be wingless.
$\exists x\big[Wx \wedge Bx \wedge \forall y\big((Wy \wedge By) \rightarrow x = y\big)\big]$
6. The Pegasus is a winged horse.
$\exists x\big[Px \wedge \forall y(Py \rightarrow x = y) \wedge Wx \wedge Hx\big]$
7. The animal in Farmer Brown's field is not a horse.
$\exists x\big[Bx \wedge \forall y(By \rightarrow x = y) \wedge \neg Hx\big]$
Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.
8. The horse in Farmer Brown's field does not have wings.
$\exists x\big[Hx \wedge Bx \wedge \forall y\big((Hy \wedge By) \rightarrow x = y\big) \wedge \neg Wx\big]$
Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.

D. In this chapter, we symbolized 'Nick is the traitor' by '$\exists x(Tx \wedge \forall y(Ty \rightarrow x = y) \wedge x = n)$'. Two equally good symbolizations would be:

- $Tn \wedge \forall y(Ty \rightarrow n = y)$

This sentence requires that Nick is a traitor, and that Nick alone is a traitor. Otherwise put, there is one and only one traitor, namely, Nick. Otherwise put: Nick is the traitor.

- $\forall y(Ty \leftrightarrow y = n)$
  This sentence can be understood thus: Take anything you like; now, if you chose a traitor, you chose Nick, and if you chose Nick, you chose a traitor. So there is one and only one traitor, namely, Nick, as required.

Explain why these would be equally good symbolizations.

CHAPTER 26

Sentences of FOL

A. Identify which variables are bound and which are free. We underline the bound variables, and overline the free variables.

1. $\exists x L\underline{x}\overline{y} \wedge \forall y L\underline{y}\overline{x}$
2. $\forall x A\underline{x} \wedge B\overline{x}$
3. $\forall x(A\underline{x} \wedge B\underline{x}) \wedge \forall y(C\overline{x} \wedge D\underline{y})$
4. $\forall x \exists y[R\underline{xy} \rightarrow (J\overline{z} \wedge K\underline{x})] \vee R\overline{yx}$
5. $\forall x_1(M\overline{x_2} \leftrightarrow L\overline{x_2}\underline{x_1}) \wedge \exists x_2 L\overline{x_3}\underline{x_2}$

CHAPTER 28

Truth in FOL

A. Consider the following interpretation:

- The domain comprises only Corwin and Benedict
- 'Ax' is to be true of both Corwin and Benedict
- 'Bx' is to be true of Benedict only
- 'Nx' is to be true of no one
- 'c' is to refer to Corwin

Determine whether each of the following sentences is true or false in that interpretation:

1. Bc False
2. $Ac \leftrightarrow \neg Nc$ True
3. $Nc \rightarrow (Ac \vee Bc)$ True
4. $\forall x Ax$ True
5. $\forall x \neg Bx$ False
6. $\exists x(Ax \wedge Bx)$ True
7. $\exists x(Ax \rightarrow Nx)$ False
8. $\forall x(Nx \vee \neg Nx)$ True
9. $\exists x Bx \rightarrow \forall x Ax$ True

B. Consider the following interpretation:

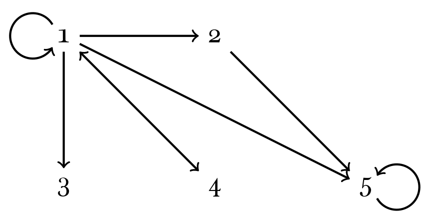
- The domain comprises only Lemmy, Courtney and Eddy
- 'Gx' is to be true of Lemmy, Courtney and Eddy.
- 'Hx' is to be true of and only of Courtney
- 'Mx' is to be true of and only of Lemmy and Eddy
- 'c' is to refer to Courtney
- 'e' is to refer to Eddy

Determine whether each of the following sentences is true or false in that interpretation:

1. Hc True
2. He False

3. $Mc \lor Me$ — True
4. $Gc \lor \neg Gc$ — True
5. $Mc \rightarrow Gc$ — True
6. $\exists x Hx$ — True
7. $\forall x Hx$ — False
8. $\exists x \neg Mx$ — True
9. $\exists x(Hx \land Gx)$ — True
10. $\exists x(Mx \land Gx)$ — True
11. $\forall x(Hx \lor Mx)$ — True
12. $\exists x Hx \land \exists x Mx$ — True
13. $\forall x(Hx \leftrightarrow \neg Mx)$ — True
14. $\exists x Gx \land \exists x \neg Gx$ — False
15. $\forall x \exists y(Gx \land Hy)$ — True

C. Following the diagram conventions introduced at the end of §23, consider the following interpretation:



Determine whether each of the following sentences is true or false in that interpretation:

1. $\exists x Rxx$ — True
2. $\forall x Rxx$ — False
3. $\exists x \forall y Rxy$ — True
4. $\exists x \forall y Ryx$ — False
5. $\forall x \forall y \forall z((Rxy \land Ryz) \rightarrow Rxz)$ — False
6. $\forall x \forall y \forall z((Rxy \land Rxz) \rightarrow Ryz)$ — False
7. $\exists x \forall y \neg Rxy$ — True
8. $\forall x(\exists y Rxy \rightarrow \exists y Ryx)$ — True
9. $\exists x \exists y(\neg x = y \land Rxy \land Ryx)$ — True
10. $\exists x \forall y(Rxy \leftrightarrow x = y)$ — True
11. $\exists x \forall y(Ryx \leftrightarrow x = y)$ — False
12. $\exists x \exists y(\neg x = y \land Rxy \land \forall z(Rzx \leftrightarrow y = z))$ — True

CHAPTER 30

Using Interpretations

A. Show that each of the following is neither a logical truth nor a contradiction:

1. $Da \wedge Db$
2. $\exists x T x h$
3. $Pm \wedge \neg \forall x P x$
4. $\forall z J z \leftrightarrow \exists y J y$
5. $\forall x(W x m n \vee \exists y L x y)$
6. $\exists x(G x \rightarrow \forall y M y)$
7. $\exists x(x = h \wedge x = i)$

B. Show that the following pairs of sentences are not logically equivalent.

1. Ja, Ka
2. $\exists x J x$, Jm
3. $\forall x R x x$, $\exists x R x x$
4. $\exists x P x \rightarrow Qc$, $\exists x(P x \rightarrow Qc)$
5. $\forall x(P x \rightarrow \neg Q x)$, $\exists x(P x \wedge \neg Q x)$
6. $\exists x(P x \wedge Q x)$, $\exists x(P x \rightarrow Q x)$
7. $\forall x(P x \rightarrow Q x)$, $\forall x(P x \wedge Q x)$
8. $\forall x \exists y R x y$, $\exists x \forall y R x y$
9. $\forall x \exists y R x y$, $\forall x \exists y R y x$

C. Show that the following sentences are jointly consistent:

1. $Ma, \neg Na, Pa, \neg Qa$
2. $Lee, Leg, \neg Lge, \neg Lgg$
3. $\neg(Ma \wedge \exists x A x), Ma \vee Fa, \forall x(F x \rightarrow A x)$
4. $Ma \vee Mb, Ma \rightarrow \forall x \neg M x$
5. $\forall y G y, \forall x(G x \rightarrow H x), \exists y \neg I y$
6. $\exists x(B x \vee A x), \forall x \neg C x, \forall x\big[(A x \wedge B x) \rightarrow C x\big]$

7. $\exists x Xx, \exists x Yx, \forall x(Xx \leftrightarrow \neg Yx)$
8. $\forall x(Px \lor Qx), \exists x \neg(Qx \land Px)$
9. $\exists z(Nz \land Ozz), \forall x \forall y(Oxy \rightarrow Oyx)$
10. $\neg \exists x \forall y Rxy, \forall x \exists y Rxy$
11. $\neg Raa, \forall x(x = a \lor Rxa)$
12. $\forall x \forall y \forall z[(x = y \lor y = z) \lor x = z], \exists x \exists y\ \neg x = y$
13. $\exists x \exists y((Zx \land Zy) \land x = y), \neg Zd, d = e$

D. Show that the following arguments are invalid:

1. $\forall x(Ax \rightarrow Bx) \therefore \exists x Bx$
2. $\forall x(Rx \rightarrow Dx), \forall x(Rx \rightarrow Fx) \therefore \exists x(Dx \land Fx)$
3. $\exists x(Px \rightarrow Qx) \therefore \exists x Px$
4. $Na \land Nb \land Nc \therefore \forall x Nx$
5. $Rde, \exists x Rxd \therefore Red$
6. $\exists x(Ex \land Fx), \exists x Fx \rightarrow \exists x Gx \therefore \exists x(Ex \land Gx)$
7. $\forall x Oxc, \forall x Ocx \therefore \forall x Oxx$
8. $\exists x(Jx \land Kx), \exists x \neg Kx, \exists x \neg Jx \therefore \exists x(\neg Jx \land \neg Kx)$
9. $Lab \rightarrow \forall x Lxb, \exists x Lxb \therefore Lbb$
10. $\forall x(Dx \rightarrow \exists y Tyx) \therefore \exists y \exists z\ \neg y = z$

CHAPTER 32

Basic rules for FOL

A. The following two 'proofs' are *incorrect*. Explain why both are incorrect. Also, provide interpretations which would invalidate the fallacious argument forms the 'proofs' enshrine:

1	$\forall x Rxx$	
2	Raa	$\forall$E 1
3	$\forall y Ray$	$\forall$I 2
4	$\forall x \forall y Rxy$	$\forall$I 3

When using $\forall$I, you must replace *all* names with the new variable. So line 3 is bogus. As a counterinterpretation, consider the following:

1	$\forall x \exists y Rxy$	
2	$\exists y Ray$	$\forall$E 1
3	Raa	
4	$\exists x Rxx$	$\exists$I 3
5	$\exists x Rxx$	$\exists$E 2, 3–4

The instantiating constant, 'a', occurs in the line (line 2) to which $\exists$E is to be applied on line 5. So the use of $\exists$E on line 5 is bogus. As a counterinterpretation, consider the following:

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs.

1	$\forall x \exists y (Rxy \lor Ryx)$	
2	$\forall x \neg Rmx$	
3	$\exists y (Rmy \lor Rym)$	$\forall$E 1
4	$Rma \lor Ram$	
5	$\neg Rma$	$\forall$E 2
6	Ram	DS 4, 5
7	$\exists x Rxm$	$\exists$I 6
8	$\exists x Rxm$	$\exists$E 3, 4–7

1	$\forall x(\exists y Lxy \to \forall z Lzx)$	
2	Lab	
3	$\exists y Lay \to \forall z Lza$	$\forall$E 1
4	$\exists y Lay$	$\exists$I 2
5	$\forall z Lza$	$\to$E 3, 4
6	Lca	$\forall$E 5
7	$\exists y Lcy \to \forall z Lzc$	$\forall$E 1
8	$\exists y Lcy$	$\exists$I 6
9	$\forall z Lzc$	$\to$E 7, 8
10	Lcc	$\forall$E 9
11	$\forall x Lxx$	$\forall$I 10

1	$\forall x(Jx \to Kx)$	
2	$\exists x \forall y Lxy$	
3	$\forall x Jx$	
4	$\quad \forall y Lay$	
5	$\quad Laa$	$\forall$E 4
6	$\quad Ja$	$\forall$E 3
7	$\quad Ja \to Ka$	$\forall$E 1
8	$\quad Ka$	$\to$E 7, 6
9	$\quad Ka \wedge Laa$	$\wedge$I 8, 5
10	$\quad \exists x(Kx \wedge Lxx)$	$\exists$I 9
11	$\exists x(Kx \wedge Lxx)$	$\exists$E 2, 4–10

C. In §22 problem A, we considered fifteen syllogistic figures of Aristotelian logic. Provide proofs for each of the argument forms. NB: You will find it *much* easier if you symbolize (for example) 'No F is G' as '$\forall x(Fx \to \neg Gx)$'.

I shall prove the four Figure I syllogisms; the rest are *extremely* similar.

Barbara

1	$\forall x(Gx \to Fx)$	
2	$\forall x(Hx \to Gx)$	
3	$Ga \to Fa$	$\forall$E 1
4	$Ha \to Ga$	$\forall$E 2
5	$\quad Ha$	
6	$\quad Ga$	$\to$E 4, 5
7	$\quad Fa$	$\to$E 3, 6
8	$Ha \to Fa$	$\to$I 5–7
9	$\forall x(Hx \to Fx)$	$\forall$I 8

Celerant is exactly as Barbara, replacing 'F' with '$\neg F$' throughout.

Ferio

1	$\forall x(Gx \to \neg Fx)$	
2	$\exists x(Hx \wedge Gx)$	
3	$\quad Ha \wedge Ga$	
4	$\quad Ha$	$\wedge$E 3
5	$\quad Ga$	$\wedge$E 3
6	$\quad Ga \to \neg Fa$	$\forall$E 1
7	$\quad \neg Fa$	$\to$E 6, 5
8	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 7
9	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 2, 3–9

Darii is exactly as Ferio, replacing '$\neg F$' with 'F' throughout.

D. Aristotle and his successors identified other syllogistic forms which depended upon 'existential import'. Symbolize each of the following argument forms in FOL and offer proofs.

- **Barbari**. Something is H. All G are F. All H are G. So: Some H is F
  $\exists x Hx, \forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge Fx)$

1	$\exists x Hx$	
2	$\forall x(Gx \rightarrow Fx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$\mid\ Ha$	
5	$\mid\ Ha \rightarrow Ga$	$\forall$E 3
6	$\mid\ Ga$	$\rightarrow$E 5, 4
7	$\mid\ Ga \rightarrow Fa$	$\forall$E 2
8	$\mid\ Fa$	$\rightarrow$E 7, 6
9	$\mid\ Ha \wedge Fa$	$\wedge$I 4, 8
10	$\mid\ \exists x(Hx \wedge Fx)$	$\exists$I 9
11	$\exists x(Hx \wedge Fx)$	$\exists$E 1, 4–10

- **Celaront**. Something is H. No G are F. All H are G. So: Some H is not F
  $\exists x Hx, \forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$
  Proof is exactly as for Barbari, replacing 'F' with '$\neg F$' throughout.
- **Cesaro**. Something is H. No F are G. All H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$\mid\ Ha$	
5	$\mid\ Ha \rightarrow Ga$	$\forall$E 3
6	$\mid\ Ga$	$\rightarrow$E 5, 4
7	$\mid\ Fa \rightarrow \neg Ga$	$\forall$E 2
8	$\mid\ \mid\ Fa$	
9	$\mid\ \mid\ \neg Ga$	$\rightarrow$E 7, 8
10	$\mid\ \mid\ \bot$	$\bot$I 6, 9
11	$\mid\ \neg Fa$	$\neg$I 8–10
12	$\mid\ Ha \wedge \neg Fa$	$\wedge$I 4, 11
13	$\mid\ \exists x(Hx \wedge \neg Fx)$	$\exists$I 12
14	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 1, 4–13

- **Camestros**. Something is H. All F are G. No H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Hx \rightarrow \neg Gx)$	
4	$\quad Ha$	
5	$\quad Ha \rightarrow \neg Ga$	$\forall$E 3
6	$\quad \neg Ga$	$\rightarrow$E 5, 4
7	$\quad Fa \rightarrow Ga$	$\forall$E 2
8	$\quad \neg Fa$	MT 7, 6
9	$\quad Ha \land \neg Fa$	$\land$I 4, 8
10	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \land \neg Fx)$	$\exists$E 1, 4–10

- **Felapton**. Something is G. No G are F. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Gx \rightarrow \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Gx \rightarrow \neg Fx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$\quad Ga$	
5	$\quad Ga \rightarrow Ha$	$\forall$E 3
6	$\quad Ha$	$\rightarrow$E 5, 4
7	$\quad Ga \rightarrow \neg Fa$	$\forall$E 2
8	$\quad \neg Fa$	$\rightarrow$E 7, 4
9	$\quad Ha \land \neg Fa$	$\land$I 6, 8
10	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

- **Darapti**. Something is G. All G are F. All G are H. So: Some H is F.
  $\exists x Gx, \forall x(Gx \rightarrow Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \land Fx)$
  Proof is exactly as for Felapton, replacing '$\neg F$' with 'F' throughout.

- **Calemos**. Something is H. All F are G. No G are H. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Gx \rightarrow \neg Hx)$	
4	$\quad Ha$	
5	$\quad Ga \rightarrow \neg Ha$	$\forall$E 3
6	$\quad\quad Ga$	
7	$\quad\quad \neg Ha$	$\rightarrow$E 5, 6
8	$\quad\quad \bot$	$\bot$I 4, 7
9	$\quad \neg Ga$	$\neg$I 6–8
10	$\quad Fa \rightarrow Ga$	$\forall$E 2
11	$\quad \neg Fa$	MT 10, 9
12	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 11
13	$\quad \exists x(Hx \wedge Fx)$	$\exists$I 12
14	$\exists x(Hx \wedge Fx)$	$\exists$E 1, 4–13

- **Fesapo**. Something is G. No F is G. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Fx \rightarrow \neg Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$\quad Ga$	
5	$\quad Ga \rightarrow Ha$	$\forall$E 3
6	$\quad Ha$	$\rightarrow$E 5, 4
7	$\quad Fa \rightarrow \neg Ga$	$\forall$E 2
8	$\quad\quad Fa$	
9	$\quad\quad \neg Ga$	$\rightarrow$E 7, 8
10	$\quad\quad \bot$	$\bot$I 4, 9
11	$\quad \neg Fa$	$\neg$I 8–10
12	$\quad Ha \wedge \neg Fa$	$\wedge$I 6, 11
13	$\quad \exists x(Hx \wedge Fx)$	$\exists$I 12
14	$\exists x(Hx \wedge Fx)$	$\exists$E 1, 4–13

- **Bamalip**. Something is F. All F are G. All G are H. So: Some H are F.
  $\exists xFx, \forall x(Fx \to Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$

1	$\exists xFx$	
2	$\forall x(Fx \to Gx)$	
3	$\underline{\forall x(Gx \to Hx)}$	
4	$\mid\ \underline{Fa}$	
5	$\mid\ Fa \to Ga$	$\forall$E 2
6	$\mid\ Ga$	$\to$E 5, 4
7	$\mid\ Ga \to Ha$	$\forall$E 3
8	$\mid\ Ha$	$\to$E 7, 6
9	$\mid\ Ha \land Fa$	$\land$I 8, 4
10	$\mid\ \exists x(Hx \land Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

E. Provide a proof of each claim.

1. $\vdash \forall xFx \lor \neg\forall xFx$

1	$\mid\ \underline{\forall xFx}$	
2	$\mid\ \forall xFx \lor \neg\forall xFx$	$\lor$I 1
3	$\mid\ \underline{\neg\forall xFx}$	
4	$\mid\ \forall xFx \lor \neg\forall xFx$	$\lor$I 3
5	$\forall xFx \lor \neg\forall xFx$	TND 1–2, 3–4

2. $\vdash \forall z(Pz \lor \neg Pz)$

1	$\mid\ \underline{Pa}$	
2	$\mid\ Pa \lor \neg Pa$	$\lor$I 1
3	$\mid\ \underline{\neg Pa}$	
4	$\mid\ Pa \lor \neg Pa$	$\lor$I 3
5	$Pa \lor \neg Pa$	TND 1–2, 3–4
6	$\forall x(Px \lor \neg Px)$	$\forall$I 5

3. $\forall x(Ax \to Bx), \exists xAx \vdash \exists xBx$

1	$\forall x(Ax \rightarrow Bx)$	
2	$\exists x Ax$	
3	$\quad Aa$	
4	$\quad Aa \rightarrow Ba$	$\forall$E 1
5	$\quad Ba$	$\rightarrow$E 4, 3
6	$\quad \exists x Bx$	$\exists$I 5
7	$\exists x Bx$	$\exists$E 2, 3–6

4. $\forall x(Mx \leftrightarrow Nx), Ma \land \exists xRxa \vdash \exists xNx$

1	$\forall x(Mx \leftrightarrow Nx)$	
2	$Ma \land \exists xRxa$	
3	Ma	$\land$E 2
4	$Ma \leftrightarrow Na$	$\forall$E 1
5	Na	$\leftrightarrow$E 4, 3
6	$\exists xNx$	$\exists$I 5

5. $\forall x\forall yGxy \vdash \exists xGxx$

1	$\forall x\forall yGxy$	
2	$\forall yGay$	$\forall$E 1
3	Gaa	$\forall$E 2
4	$\exists xGxx$	$\exists$I 3

6. $\vdash \forall xRxx \to \exists x\exists yRxy$

1	$\quad \forall xRxx$	
2	$\quad Raa$	$\forall$E 1
3	$\quad \exists yRay$	$\exists$I 2
4	$\quad \exists x\exists yRxy$	$\exists$I 3
5	$\forall xRxx \to \exists x\exists yRxy$	$\to$I 1–4

7. $\vdash \forall y\exists x(Qy \to Qx)$

1	$\quad Qa$	
2	$\quad Qa$	R 1
3	$Qa \to Qa$	$\to$I 1–2
4	$\exists x(Qa \to Qx)$	$\exists$I 3
5	$\forall y\exists x(Qy \to Qx)$	$\forall$I 4

8. $Na \to \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb \vdash \neg Na$

1	$Na \to \forall x(Mx \leftrightarrow Ma)$	
2	Ma	
3	$\neg Mb$	
4	$\quad Na$	
5	$\quad \forall x(Mx \leftrightarrow Ma)$	$\to$E 1, 4
6	$\quad Mb \leftrightarrow Ma$	$\forall$E 5
7	$\quad Mb$	$\leftrightarrow$E 6, 2
8	$\quad \bot$	$\bot$I 7, 3
9	$\neg Na$	$\neg$I 4–8

9. $\forall x \forall y(Gxy \rightarrow Gyx) \vdash \forall x \forall y(Gxy \leftrightarrow Gyx)$

1	$\forall x \forall y(Gxy \rightarrow Gyx)$	
2	$\quad Gab$	
3	$\quad \forall y(Gay \rightarrow Gya)$	$\forall$E 1
4	$\quad Gab \rightarrow Gba$	$\forall$E 3
5	$\quad Gba$	$\rightarrow$E 4, 2
6	$\quad Gba$	
7	$\quad \forall y(Gby \rightarrow Gyb)$	$\forall$E 1
8	$\quad Gba \rightarrow Gab$	$\forall$E 7
9	$\quad Gab$	$\rightarrow$E 8, 6
10	$Gab \leftrightarrow Gba$	$\leftrightarrow$I 2–5, 6–9
11	$\forall y(Gay \leftrightarrow Gya)$	$\forall$I 10
12	$\forall x \forall y(Gxy \leftrightarrow Gyx)$	$\forall$I 11

10. $\forall x(\neg Mx \vee Ljx), \forall x(Bx \rightarrow Ljx), \forall x(Mx \vee Bx) \vdash \forall x Ljx$

1	$\forall x(\neg Mx \vee Ljx)$	
2	$\forall x(Bx \rightarrow Ljx)$	
3	$\forall x(Mx \vee Bx)$	
4	$\neg Ma \vee Ljx$	$\forall$E 1
5	$Ba \rightarrow Lja$	$\forall$E 2
6	$Ma \vee Ba$	$\forall$E 3
7	$\quad \neg Ma$	
8	$\quad Ba$	DS 6, 7
9	$\quad Lja$	$\rightarrow$E 5, 8
10	$\quad Lja$	
11	$\quad Lja$	R 10
12	Lja	$\vee$E 4, 7–9, 10–11
13	$\forall x Ljx$	$\forall$I 12

F. Write a symbolization key for the following argument, symbolize it, and prove it:

> There is someone who likes everyone who likes everyone that she likes. Therefore, there is someone who likes herself.

Symbolization key:

domain: all people
Lxy: _____$_x$ likes _____$_y$

$\exists x \forall y(\forall z(Lxz \to Lyz) \to Lxy) \therefore \exists x Lxx$

1	$\exists x \forall y(\forall z(Lxz \to Lyz) \to Lxy)$	
2	$\quad \forall y(\forall z(Laz \to Lyz) \to Lay)$	
3	$\quad \forall z(Laz \to Laz) \to Laa$	$\forall$E 2
4	$\quad\quad Lac$	
5	$\quad\quad Lac$	R 4
6	$\quad Lac \to Lac$	$\to$I 4–5
7	$\quad \forall z(Laz \to Laz)$	$\forall$I 6
8	$\quad Laa$	$\to$E 3, 7
9	$\quad \exists x Lxx$	$\exists$I 8
10	$\exists x Lxx$	$\exists$E 1, 2—9

G. Show that each pair of sentences is provably equivalent.

1. $\forall x(Ax \to \neg Bx)$, $\neg \exists x(Ax \land Bx)$
2. $\forall x(\neg Ax \to Bd)$, $\forall x Ax \lor Bd$
3. $\exists x Px \to Qc$, $\forall x(Px \to Qc)$

H. For each of the following pairs of sentences: If they are provably equivalent, give proofs to show this. If they are not, construct an interpretation to show that they are not logically equivalent.

1. $\forall x Px \to Qc, \forall x(Px \to Qc)$ Not logically equivalent
   Counter-interpretation: let the domain be the numbers 1 and 2. Let 'c' name 1. Let 'Px' be true of and only of 1. Let 'Qx' be true of, and only of, 2.
2. $\forall x \forall y \forall z Bxyz, \forall x Bxxx$ Not logically equivalent
   Counter-interpretation: let the domain be the numbers 1 and 2. Let '$Bxyz$' be true of, and only of, $\langle 1,1,1 \rangle$ and $\langle 2,2,2 \rangle$.
3. $\forall x \forall y Dxy, \forall y \forall x Dxy$ Provably equivalent

1	$\forall x \forall y Dxy$	
2	$\forall y Day$	$\forall$E 1
3	Dab	$\forall$E 2
4	$\forall x Dxb$	$\forall$I 3
5	$\forall y \forall x Dxy$	$\forall$I 4

1	$\forall y \forall x Dxy$	
2	$\forall x Dxa$	$\forall$E 1
3	Dba	$\forall$E 2
4	$\forall y Dby$	$\forall$I 3
5	$\forall x \forall y Dxy$	$\forall$I 4

4. $\exists x \forall y Dxy, \forall y \exists x Dxy$ — Not logically equivalent
Counter-interpretation: let the domain be the numbers 1 and 2. Let 'Dxy' hold of and only of ⟨1,2⟩ and ⟨2,1⟩. This is depicted thus:

$$1 \longleftrightarrow 2$$

5. $\forall x(Rca \leftrightarrow Rxa), Rca \leftrightarrow \forall x Rxa$ — Not logically equivalent
Counter-interpretation, consider the following diagram, allowing 'a' to name 1 and 'c' to name 2:

I. For each of the following arguments: If it is valid in FOL, give a proof. If it is invalid, construct an interpretation to show that it is invalid.

1. $\exists y \forall x Rxy \therefore \forall x \exists y Rxy$ — Valid

1	$\exists y \forall x Rxy$	
2	$\quad \forall x Rxa$	
3	$\quad Rba$	∀E 2
4	$\quad \exists y Rby$	∃I 3
5	$\exists y Rby$	∃E 1, 2–4
6	$\forall x \exists y Rxy$	∀I 5

2. $\forall x \exists y Rxy \therefore \exists y \forall x Rxy$ — Not valid
Counter interpretation: let the domain be the numbers 1 and 2. Let 'Rxy' be true of 1 and 2, and of 2 and 1 (but not 1 and itself or 2 and itself).
3. $\exists x(Px \wedge \neg Qx) \therefore \forall x(Px \rightarrow \neg Qx)$ — Not valid
Counter interpretation: let the domain be the numbers 1 and 2. Let 'Px' be true of everything in the domain. Let 'Qx' be true of, and only of, 2.
4. $\forall x(Sx \rightarrow Ta), Sd \therefore Ta$ — Valid

1	$\forall x(Sx \rightarrow Ta)$	
2	Sd	
3	$Sd \rightarrow Ta$	∀E 1
4	Ta	→E 3, 2

5. $\forall x(Ax \rightarrow Bx), \forall x(Bx \rightarrow Cx) \therefore \forall x(Ax \rightarrow Cx)$ — Valid

1	$\forall x(Ax \to Bx)$	
2	$\forall x(Bx \to Cx)$	
3	$Aa \to Ba$	$\forall$E 1
4	$Ba \to Ca$	$\forall$E 2
5	$\quad Aa$	
6	$\quad Ba$	$\to$E 3, 5
7	$\quad Ca$	$\to$E 4, 6
8	$Aa \to Ca$	$\to$I 5–7
9	$\forall x(Ax \to Cx)$	$\forall$I 8

6. $\exists x(Dx \lor Ex), \forall x(Dx \to Fx)$ $\therefore$ $\exists x(Dx \land Fx)$ Invalid
   Counter-interpretation: let the domain be the number 1 . Let 'Dx' hold of nothing. Let both 'Ex' and 'Fx' hold of everything.
7. $\forall x \forall y(Rxy \lor Ryx)$ $\therefore$ Rjj Valid

1	$\forall x \forall y(Rxy \lor Ryx)$	
2	$\forall y(Rjy \lor Ryj)$	$\forall$E 1
3	$Rjj \lor Rjj$	$\forall$E 2
4	$\quad Rjj$	
5	$\quad Rjj$	R 4
6	$\quad Rjj$	
7	$\quad Rjj$	R 6
8	Rjj	$\lor$E 3, 4–5, 6–7

8. $\exists x \exists y(Rxy \lor Ryx)$ $\therefore$ Rjj Invalid
   Counter-interpretation: consider the following diagram, allowing 'j' to name 2.

9. $\forall x Px \to \forall x Qx, \exists x \neg Px$ $\therefore$ $\exists x \neg Qx$ Invalid
   Counter-interpretation: let the domain be the number 1. Let 'Px' be true of nothing. Let 'Qx' be true of everything.
10. $\exists x Mx \to \exists x Nx$, $\neg \exists x Nx$ $\therefore$ $\forall x \neg Mx$ Valid

1	$\exists x Mx \to \exists x Nx$	
2	$\neg \exists x Nx$	
3	$\quad Ma$	
4	$\quad \exists x Mx$	$\exists$I 3
5	$\quad \exists x Nx$	$\to$E 1, 4
6	$\quad \bot$	$\bot$I 5, 2
7	$\neg Ma$	$\neg$I 3–6
8	$\forall x \neg Mx$	$\forall$I 7

CHAPTER 33

Conversion of quantifiers

A. Show in each case that the sentences are provably inconsistent:

1. $Sa \to Tm, Tm \to Sa, Tm \land \neg Sa$

1	$Sa \to Tm$	
2	$Tm \to Sa$	
3	$Tm \land \neg Sa$	
4	Tm	$\land$E 3
5	$\neg Sa$	$\land$E 3
6	Sa	$\to$E 2, 4
7	$\bot$	$\bot$I 5, 6

2. $\neg\exists x Rxa, \forall x \forall y Ryx$

1	$\neg\exists x Rxa$	
2	$\forall x \forall y Ryx$	
3	$\forall x \neg Rxa$	CQ 1
4	$\neg Rba$	$\forall$E 3
5	$\forall y Rya$	$\forall$E 2
6	Rba	$\forall$E 5
7	$\bot$	$\bot$I 6, 4

3. $\neg\exists x \exists y Lxy, Laa$

1	$\neg\exists x \exists y Lxy$	
2	Laa	
3	$\forall x \neg\exists y Lxy$	CQ 1
4	$\neg\exists y Lay$	$\forall$E 3
5	$\forall y \neg Lay$	CQ 4
6	$\neg Laa$	$\forall$E 5
7	$\bot$	$\bot$I 2, 6

4. $\forall x(Px \to Qx), \forall z(Pz \to Rz), \forall y Py, \neg Qa \wedge \neg Rb$

1	$\forall x(Px \to Qx)$	
2	$\forall z(Pz \to Rz)$	
3	$\forall y Py$	
4	$\neg Qa \wedge \neg Rb$	
5	$\neg Qa$	$\wedge$E 4
6	$Pa \to Qa$	$\forall$E 1
7	$\neg Pa$	MT 6, 5
8	Pa	$\forall$E 3
9	$\bot$	$\bot$I 8, 7

B. Show that each pair of sentences is provably equivalent:

1. $\forall x(Ax \to \neg Bx), \neg\exists x(Ax \wedge Bx)$

1	$\forall x(Ax \to \neg Bx)$	
2	$\quad \exists x(Ax \wedge Bx)$	
3	$\quad\quad Aa \wedge Ba$	
4	$\quad\quad Aa$	$\wedge$E 3
5	$\quad\quad Ba$	$\wedge$E 3
6	$\quad\quad Aa \to \neg Ba$	$\forall$E 1
7	$\quad\quad \neg Ba$	$\to$E 6, 4
8	$\quad\quad \bot$	$\bot$I 5, 7
9	$\quad \bot$	$\exists$E 2, 3–8
10	$\neg\exists x(Ax \wedge Bx)$	$\neg$I 2–9

1	$\neg\exists x(Ax \wedge Bx)$	
2	$\forall x \neg(Ax \wedge Bx)$	CQ 1
3	$\neg(Aa \wedge Ba)$	$\forall$E 2
4	$\quad Aa$	
5	$\quad\quad Ba$	
6	$\quad\quad Aa \wedge Ba$	$\wedge$I 4, 5
7	$\quad\quad \bot$	$\bot$I 6, 3
8	$\quad \neg Ba$	$\neg$I 5–7
9	$Aa \to \neg Ba$	$\to$I 4–8
10	$\forall x(Ax \to \neg Bx)$	$\forall$I 9

2. $\forall x(\neg Ax \to Bd), \forall x Ax \vee Bd$

1	$\forall x(\neg Ax \to Bd)$	
2	$\neg Aa \to Bd$	$\forall$E 1
3	Bd	
4	$\forall xAx \lor Bd$	$\lor$I 6
5	$\neg Bd$	
6	$\neg\neg Aa$	MT 2, 5
7	Aa	DNE 6
8	$\forall xAx$	$\forall$E 7
9	$\forall xAx \lor Bd$	$\lor$I 8
10	$\forall xAx \lor Bd$	TND 3–4, 5–9

1	$\forall xAx \lor Bd$	
2	$\neg Aa$	
3	$\forall xAx$	
4	Aa	$\forall$E 3
5	$\bot$	$\bot$I 4, 2
6	$\neg\forall xAx$	$\neg$I 3–5
7	Bd	DS 1, 6
8	$\neg Aa \to Bd$	$\to$I 2–7
9	$\forall x(Ax \to Bd)$	$\forall$I 8

C. In §22, we considered what happens when we move quantifiers 'across' various logical operators. Show that each pair of sentences is provably equivalent:

1. $\forall x(Fx \land Ga), \forall xFx \land Ga$

1	$\forall x(Fx \land Ga)$	
2	$Fb \land Ga$	$\forall$E 1
3	Fb	$\land$E 2
4	Ga	$\land$E 6
5	$\forall xFx$	$\forall$I 3
6	$\forall xFx \land Ga$	$\land$I 5, 4

1	$\forall xFx \land Ga$	
2	$\forall xFx$	$\land$E 1
3	Ga	$\land$E 1
4	Fb	$\forall$E 2
5	$Fb \land Ga$	$\land$I 4, 3
6	$\forall x(Fx \land Ga)$	$\forall$I 5

2. $\exists x(Fx \lor Ga), \exists xFx \lor Ga$

1	$\exists x(Fx \lor Ga)$	
2	$Fb \lor Ga$	
3	Fb	
4	$\exists xFx$	$\exists$I 3
5	$\exists xFx \lor Ga$	$\lor$I 4
6	Ga	
7	$\exists xFx \lor Ga$	$\lor$I 6
8	$\exists xFx \lor Ga$	$\lor$E 2, 3–5, 6–7
9	$\exists xFx \lor Ga$	$\exists$E 1, 2–8

1	$\exists xFx \lor Ga$	
2	$\exists xFx$	
3	Fb	
4	$Fb \lor Ga$	$\lor$I 3
5	$\exists x(Fx \lor Ga)$	$\exists$I 4
6	$\exists x(Fx \lor Ga)$	$\exists$E 2, 3–5
7	Ga	
8	$Fb \lor Ga$	$\lor$I 7
9	$\exists x(Fx \lor Ga)$	$\exists$I 8
10	$\exists x(Fx \lor Ga)$	$\lor$E 1, 2–6, 7–9

3. $\forall x(Ga \to Fx), Ga \to \forall xFx$

1	$\forall x(Ga \to Fx)$	
2	$Ga \to Fb$	$\forall$E 1
3	$\quad Ga$	
4	$\quad Fb$	$\to$E 2, 3
5	$\quad \forall xFx$	$\forall$I 4
6	$Ga \to \forall xFx$	$\to$I 3–5

1	$Ga \to \forall xFx$	
2	$\quad Ga$	
3	$\quad \forall xFx$	$\to$E 1, 2
4	$\quad Fb$	$\forall$E 3
5	$Ga \to Fb$	$\to$I 2–4
6	$\forall x(Ga \to Fx)$	$\forall$I 5

4. $\forall x(Fx \to Ga), \exists xFx \to Ga$

1	$\forall x(Fx \to Ga)$	
2	$\quad \exists xFx$	
3	$\quad\quad Fb$	
4	$\quad\quad Fb \to Ga$	$\forall$E 1
5	$\quad\quad Ga$	$\to$E 4, 3
6	$\quad Ga$	$\exists$E 2, 3–5
7	$\exists xFx \to Ga$	$\to$I 2–6

1	$\exists xFx \to Ga$	
2	$\quad Fb$	
3	$\quad \exists xFx$	$\exists$I 2
4	$\quad Ga$	$\to$E 1, 3
5	$Fb \to Ga$	$\to$I 2–4
6	$\forall x(Fx \to Ga)$	$\forall$I 5

5. $\exists x(Ga \to Fx), Ga \to \exists xFx$

1	$\exists x(Ga \to Fx)$	
2	$\quad Ga$	
3	$\quad\quad Ga \to Fb$	
4	$\quad\quad Fb$	$\to$E 3, 2
5	$\quad\quad \exists xFx$	$\exists$I 4
6	$\quad \exists xFx$	$\exists$E 1, 3–5
7	$Ga \to \exists xFx$	$\to$I 2–6

1	$Ga \to \exists xFx$	
2	$\quad Ga$	
3	$\quad \exists xFx$	
4	$\quad\quad Fb$	
5	$\quad\quad\quad Ga$	
6	$\quad\quad\quad Fb$	R 4
7	$\quad\quad Ga \to Fb$	$\to$I 5–6
8	$\quad\quad \exists x(Ga \to Fx)$	$\exists$I 7
9	$\quad \exists x(Ga \to Fx)$	$\exists$E 3, 4–8
10	$\quad \neg Ga$	
11	$\quad\quad Ga$	
12	$\quad\quad \bot$	$\bot$I 11, 10
13	$\quad\quad Fb$	$\bot$E 12
14	$\quad Ga \to Fb$	$\to$E 11–13
15	$\quad \exists x(Ga \to Fx)$	$\exists$I 14
16	$\exists x(Ga \to Fx)$	TND 2–9, 10–15

6. $\exists x(Fx \to Ga), \forall x Fx \to Ga$

1	$\mid \exists x(Fx \to Ga)$	
2	$\mid \quad \mid \forall x Fx$	
3	$\mid \quad \mid \quad \mid Fb \to Ga$	
4	$\mid \quad \mid \quad \mid Fb$	$\forall$E 2
5	$\mid \quad \mid \quad \mid Ga$	$\to$E 3, 4
6	$\mid \quad \mid Ga$	$\exists$E 1, 3–5
7	$\mid \forall x Fx \to Ga$	$\to$I 2–6

1	$\mid \forall x Fx \to Ga$	
2	$\mid \quad \mid \forall x Fx$	
3	$\mid \quad \mid Ga$	$\to$E 1, 2
4	$\mid \quad \mid \quad \mid Fb$	
5	$\mid \quad \mid \quad \mid Ga$	R 3
6	$\mid \quad \mid Fb \to Ga$	$\to$I 4–5
7	$\mid \quad \mid \exists x(Fx \to Ga)$	$\exists$I 6
8	$\mid \quad \mid \neg\forall x Fx$	
9	$\mid \quad \mid \exists x \neg Fx$	CQ 8
10	$\mid \quad \mid \quad \mid \neg Fb$	
11	$\mid \quad \mid \quad \mid \quad \mid Fb$	
12	$\mid \quad \mid \quad \mid \quad \mid \bot$	$\bot$I 11, 10
13	$\mid \quad \mid \quad \mid \quad \mid Ga$	$\bot$E 12
14	$\mid \quad \mid \quad \mid Fb \to Ga$	$\to$I 11–13
15	$\mid \quad \mid \quad \mid \exists x(Fx \to Ga)$	$\exists$I 14
16	$\mid \quad \mid \exists x(Fx \to Ga)$	$\exists$E 9, 10–15
17	$\mid \exists x(Fx \to Ga)$	TND 2–7, 8–16

NB: the variable 'x' does not occur in 'Ga'.

When all the quantifiers occur at the beginning of a sentence, that sentence is said to be in *prenex normal form*. Together with the CQ rules, these equivalences are sometimes called *prenexing rules*, since they give us a means for putting any sentence into prenex normal form.

CHAPTER 34

Rules for identity

A. Provide a proof of each claim.

1. $Pa \lor Qb, Qb \to b = c, \neg Pa \vdash Qc$

1	$Pa \lor Qb$	
2	$Qb \to b = c$	
3	$\neg Pa$	
4	Qb	DS 1, 3
5	$b = c$	$\to$E 2, 4
6	Qc	=E 5, 4

2. $m = n \lor n = o, An \vdash Am \lor Ao$

1	$m = n \lor n = o$	
2	An	
3	$m = n$	
4	Am	=E 3, 2
5	$Am \lor Ao$	$\lor$I 4
6	$n = o$	
7	Ao	=E 6, 7
8	$Am \lor Ao$	$\lor$I 7
9	$Am \lor Ao$	$\lor$E 1, 3–5, 6–8

3. $\forall x\, x = m, Rma \vdash \exists x Rxx$

1	$\forall x\, x = m$	
2	Rma	
3	$a = m$	$\forall$E 1
4	Raa	=E 3, 2
5	$\exists x Rxx$	$\exists$I 4

4. $\forall x \forall y(Rxy \rightarrow x = y) \vdash Rab \rightarrow Rba$

1	$\forall x \forall y(Rxy \rightarrow x = y)$	
2	$\quad Rab$	
3	$\quad \forall y(Ray \rightarrow a = y)$	$\forall$E 1
4	$\quad Rab \rightarrow a = b$	$\forall$E 3
5	$\quad a = b$	$\rightarrow$E 4, 2
6	$\quad Raa$	=E 5, 2
7	$\quad Rba$	=E 5, 6
8	$Rab \rightarrow Rba$	$\rightarrow$I 2–7

5. $\neg\exists x \neg x = m \vdash \forall x \forall y(Px \rightarrow Py)$

1	$\neg\exists x \neg x = m$	
2	$\forall x \neg\neg x = m$	CQ 1
3	$\neg\neg a = m$	$\forall$E 2
4	$a = m$	DNE 3
5	$\neg\neg b = m$	$\forall$E 2
6	$b = m$	DNE 5
7	$\quad Pa$	
8	$\quad Pm$	=E 3, 7
9	$\quad Pb$	=E 5, 8
10	$Pa \rightarrow Pb$	$\rightarrow$I 7–9
11	$\forall y(Pa \rightarrow Py)$	$\forall$I 10
12	$\forall x \forall y(Px \rightarrow Py)$	$\forall$I 11

6. $\exists x Jx, \exists x \neg Jx \vdash \exists x \exists y\, \neg x = y$

1	$\exists x Jx$	
2	$\exists x \neg Jx$	
3	$\quad Ja$	
4	$\quad\quad \neg Jb$	
5	$\quad\quad\quad a = b$	
6	$\quad\quad\quad Jb$	=E 5, 3
7	$\quad\quad\quad \bot$	⊥I 6, 4
8	$\quad\quad \neg a = b$	¬I 5–7
9	$\quad\quad \exists y \neg a = y$	∃I 8
10	$\quad\quad \exists x \exists y \neg x = y$	∃I 9
11	$\quad \exists x \exists y \neg x = y$	∃E 2, 4–10
12	$\exists x \exists y \neg x = y$	∃E 1, 3–11

7. $\forall x(x = n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$

1	$\forall x(x = n \leftrightarrow Mx)$	
2	$\forall x(Ox \vee \neg Mx)$	
3	$n = n \leftrightarrow Mn$	∀E 1
4	$n = n$	=I
5	Mn	↔E 3, 4
6	$On \vee \neg Mn$	∀E 2
7	$\quad \neg On$	
8	$\quad \neg Mn$	DS 6, 7
9	$\quad \bot$	⊥I 5, 8
10	$\neg\neg On$	¬I 7–9
11	On	DNE 10

8. $\exists x Dx, \forall x(x = p \leftrightarrow Dx) \vdash Dp$

1	$\exists x Dx$	
2	$\forall x(x = p \leftrightarrow Dx)$	
3	$\quad Dc$	
4	$\quad c = p \leftrightarrow Dc$	∀E 2
5	$\quad c = p$	↔E 4, 3
6	$\quad Dp$	=E 5, 3
7	Dp	∃E 1, 3–6

9. $\exists x\big[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx\big], Kd \vdash Bd$

1	$\exists x\big[(Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Bx\big]$	
2	Kd	
3	$(Ka \wedge \forall y(Ky \rightarrow a = y)) \wedge Ba$	
4	$Ka \wedge \forall y(Ky \rightarrow a = y)$	$\wedge$E 3
5	Ka	$\wedge$E 4
6	$\forall y(Ky \rightarrow a = y)$	$\wedge$E 4
7	$Kd \rightarrow a = d$	$\forall$E 6
8	$a = d$	$\rightarrow$E 7, 2
9	Ba	$\wedge$E 3
10	Bd	=E 8, 9
11	Bd	$\exists$E 1, 3–10

10. $\vdash Pa \rightarrow \forall x(Px \vee \neg x = a)$

1	Pa	
2	$b = a$	
3	Pb	=E 2, 1
4	$Pb \vee \neg b = a$	$\vee$I 3
5	$\neg b = a$	
6	$Pb \vee \neg b = a$	$\vee$I 5
7	$Pb \vee \neg b = a$	TND 2–4, 5–6
8	$\forall x(Px \vee \neg x = a)$	$\forall$I 7
9	$Pa \rightarrow \forall x(Px \vee \neg x = a)$	$\rightarrow$I 1–8

B. Show that the following are provably equivalent:

- $\exists x\big([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n\big)$
- $Fn \wedge \forall y(Fy \rightarrow n = y)$

And hence that both have a decent claim to symbolize the English sentence 'Nick is the F'.

In one direction:

1	$\exists x\big([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n\big)$	
2	$\quad [Fa \wedge \forall y(Fy \rightarrow a = y)] \wedge a = n$	
3	$\quad a = n$	$\wedge$E 2
4	$\quad Fa \wedge \forall y(Fy \rightarrow a = y)$	$\wedge$E 2
5	$\quad Fa$	$\wedge$E 4
6	$\quad Fn$	=E 3, 5
7	$\quad \forall y(Fy \rightarrow a = y)$	$\wedge$E 4
8	$\quad \forall y(Fy \rightarrow n = y)$	=E 3, 7
9	$\quad Fn \wedge \forall y(Fy \rightarrow n = y)$	$\wedge$I 6, 8
10	$Fn \wedge \forall y(Fy \rightarrow n = y)$	$\exists$E 1, 2–9

And now in the other:

1	$Fn \wedge \forall y(Fy \rightarrow n = y)$	
2	$n = n$	=I
3	$[Fn \wedge \forall y(Fy \rightarrow n = y)] \wedge n = n$	$\wedge$I 1, 2
4	$\exists x\big([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n\big)$	$\exists$I 3

C. In §24, we claimed that the following are logically equivalent symbolizations of the English sentence 'there is exactly one F':

- $\exists xFx \wedge \forall x\forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$
- $\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$
- $\exists x\forall y(Fy \leftrightarrow x = y)$

Show that they are all provably equivalent. (*Hint*: to show that three claims are provably equivalent, it suffices to show that the first proves the second, the second proves the third and the third proves the first; think about why.)

It suffices to show that the first proves the second, the second proves the third and the third proves the first, for we can then show that any of them prove any others, just by chaining the proofs together (numbering lines, where necessary. Armed with this, we start on the first proof:

1	$\exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \to x = y\big]$	
2	$\exists x Fx$	$\wedge$E 1
3	$\forall x \forall y\big[(Fx \wedge Fy) \to x = y\big]$	$\wedge$E 1
4	$\quad \mid\ Fa$	
5	$\quad \mid\ \forall y\big[(Fa \wedge Fy) \to a = y\big]$	$\forall$E 3
6	$\quad \mid\ (Fa \wedge Fb) \to a = b$	$\forall$E 5
7	$\quad \mid\ \quad \mid\ Fb$	
8	$\quad \mid\ \quad \mid\ Fa \wedge Fb$	$\wedge$I 4, 7
9	$\quad \mid\ \quad \mid\ a = b$	$\to$E 6, 8
10	$\quad \mid\ Fb \to a = b$	$\to$I 7–9
11	$\quad \mid\ \forall y(Fy \to a = y)$	$\forall$I 10
12	$\quad \mid\ Fa \wedge \forall y(Fy \to a = y))$	$\wedge$I 4, 11
13	$\quad \mid\ \exists x\big[Fx \wedge \forall y(Fy \to x = y)\big]$	$\exists$I 12
14	$\exists x\big[Fx \wedge \forall y(Fy \to x = y)\big]$	$\exists$E 2, 4–13

Now for the second proof:

1	$\exists x\big[Fx \wedge \forall y(Fy \to x = y)\big]$	
2	$\quad \mid\ Fa \wedge \forall y(Fy \to a = y)$	
3	$\quad \mid\ Fa$	$\wedge$E 2
4	$\quad \mid\ \forall y(Fy \to a = y)$	$\wedge$E 2
5	$\quad \mid\ \quad \mid\ Fb$	
6	$\quad \mid\ \quad \mid\ Fb \to a = b$	$\forall$E 4
7	$\quad \mid\ \quad \mid\ a = b$	$\to$E 6, 5
8	$\quad \mid\ \quad \mid\ a = b$	
9	$\quad \mid\ \quad \mid\ Fb$	=E 8, 3
10	$\quad \mid\ Fb \leftrightarrow a = b$	$\leftrightarrow$I 5–7, 8–9
11	$\quad \mid\ \forall y(Fy \leftrightarrow a = y)$	$\forall$I 10
12	$\quad \mid\ \exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$I 11
13	$\exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$E 1, 2–12

And finally, the third proof:

1	$\exists x \forall y(Fy \leftrightarrow x = y)$	
2	$\quad \forall y(Fy \leftrightarrow a = y)$	
3	$\quad Fa \leftrightarrow a = a$	$\forall$E 2
4	$\quad a = a$	=I
5	$\quad Fa$	$\leftrightarrow$E 3, 4
6	$\quad \exists x Fx$	$\exists$I 5
7	$\quad\quad Fb \wedge Fc$	
8	$\quad\quad Fb$	$\wedge$E 7
9	$\quad\quad Fb \leftrightarrow a = b$	$\forall$E 2
10	$\quad\quad a = b$	$\leftrightarrow$E 9, 8
11	$\quad\quad Fc$	$\wedge$E 7
12	$\quad\quad Fc \leftrightarrow a = c$	$\forall$E 2
13	$\quad\quad a = c$	$\leftrightarrow$E 12, 11
14	$\quad\quad b = c$	=E 10, 13
15	$\quad (Fb \wedge Fc) \rightarrow b = c$	$\rightarrow$I 8–14
16	$\quad \forall y\big[(Fb \wedge Fy) \rightarrow b = y\big]$	$\forall$I 15
17	$\quad \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\forall$I 16
18	$\quad \exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\wedge$I 6, 17
19	$\exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\exists$E 1, 2–18

D. Symbolize the following argument

> There is exactly one F. There is exactly one G. Nothing is both F and G. So: there are exactly two things that are either F or G.

And offer a proof of it.

1. $\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$
2. $\exists x\big[Gx \wedge \forall y(Gy \rightarrow x = y)\big]$
3. $\forall x(\neg Fx \vee \neg Gx)$ $\therefore$

$\therefore$ $\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$

1	$\exists x\big[Fx \land \forall y(Fy \rightarrow x = y)\big]$	
2	$\exists x\big[Gx \land \forall y(Gy \rightarrow x = y)\big]$	
3	$\forall x(\neg Fx \lor \neg Gx)$	
4	$Fa \land \forall y(Fy \rightarrow a = y)$	
5	Fa	$\land$E 4
6	$\forall y(Fy \rightarrow a = y)$	$\land$E 4
7	$\neg Fa \lor \neg Ga$	$\forall$E 3
8	$\neg Ga$	DS 7, 5
9	$Gb \land \forall y(Gy \rightarrow b = y)$	
10	Gb	$\land$E 9
11	$\forall y(Gy \rightarrow b = y)$	$\land$E 9
12	$a = b$	
13	Ga	=E 12, 10
14	$\bot$	$\bot$I 13, 8
15	$\neg a = b$	$\neg$I 12–14
16	$Fc \lor Gc$	
17	Fc	
18	$Fc \rightarrow a = c$	$\forall$E 6
19	$a = c$	$\rightarrow$E 18, 17
20	$a = c \lor b = c$	$\lor$I 19
21	Gc	
22	$Gc \rightarrow b = c$	$\forall$E 11
23	$b = c$	$\rightarrow$E 22, 21
24	$a = c \lor b = c$	$\lor$I 23
25	$a = c \lor b = c$	$\lor$E 16, 17–20, 21–24
26	$(Fc \lor Gc) \rightarrow (a = c \lor b = c)$	$\rightarrow$I 16–25
27	$\forall z((Fz \lor Gz) \rightarrow (a = z \lor b = z))$	$\forall$I 26
28	$\neg a = b \land \forall z((Fz \lor Gz) \rightarrow (a = z \lor b = z))$	$\land$I 15, 27
29	$\exists y\big[\neg a = y \land \forall z((Fz \lor Gz) \rightarrow (a = z \lor y = z))\big]$	$\exists$I 28
30	$\exists x\exists y\big[\neg x = y \land \forall z((Fz \lor Gz) \rightarrow (x = z \lor y = z))\big]$	$\exists$I 29
31	$\exists x\exists y\big[\neg x = y \land \forall z((Fz \lor Gz) \rightarrow (x = z \lor y = z))\big]$	$\exists$E 2, 9–30
32	$\exists x\exists y\big[\neg x = y \land \forall z((Fz \lor Gz) \rightarrow (x = z \lor y = z))\big]$	$\exists$E 1, 4–31

CHAPTER 35

Derived rules

A. Offer proofs which justify the addition of the second and fourth CQ rules as derived rules.

Justification for the second rule:

1	$\mid \neg\exists x Ax$	
2	$\mid \quad \mid Aa$	
3	$\mid \quad \mid \exists x Ax$	$\exists$I 2
4	$\mid \quad \mid \bot$	$\bot$I 3, 1
5	$\mid \neg Aa$	$\neg$I 2–4
6	$\mid \forall x \neg Ax$	$\forall$I 5

The fourth rule is harder to justify. Here is a proof that is relatively straightforward, but uses the derived rule DNE:

1	$\mid \neg\forall x Ax$	
2	$\mid \quad \mid \neg\exists x \neg Ax$	
3	$\mid \quad \mid \quad \mid \neg Aa$	
4	$\mid \quad \mid \quad \mid \exists x \neg Ax$	$\exists$I 3
5	$\mid \quad \mid \quad \mid \bot$	$\bot$I 4, 2
6	$\mid \quad \mid \neg\neg Aa$	$\neg$I 3–5
7	$\mid \quad \mid Aa$	DNE 6
8	$\mid \quad \mid \forall x Ax$	$\forall$I 7
9	$\mid \quad \mid \bot$	$\bot$I 8, 1
10	$\mid \neg\neg\exists x \neg Ax$	$\neg$I 2–9
11	$\mid \exists x \neg Ax$	DNE 10

And here is a proof that does not use any derived rules:

1	$\neg\forall x Ax$	
2	$\exists x \neg Ax$	
3	$\exists x \neg Ax \wedge \exists x \neg Ax$	$\wedge$I 2
4	$\exists x \neg Ax$	$\wedge$E 3
5	$\neg\exists x \neg Ax$	
6	Ab	
7	$Ab \wedge Ab$	$\wedge$I 6
8	Ab	$\wedge$E 7
9	$\neg Ab$	
10	$\exists x \neg Ax$	$\exists$I 9
11	$\bot$	$\bot$I 10, 5
12	Ab	$\bot$E 11
13	Ab	TND 6–8, 9–12
14	$\forall x Ax$	$\forall$I 13
15	$\bot$	$\bot$I 14, 1
16	$\exists x \neg Ax$	$\bot$E 15
17	$\exists x \neg Ax$	TND 2–4, 5–16

CHAPTER 37

Normal Forms and Expressive Completeness

Practice exercises

A. Consider the following sentences:

1. $(A \to \neg B)$
2. $\neg(A \leftrightarrow B)$
3. $(\neg A \lor \neg(A \land B))$
4. $(\neg(A \to B) \land (A \to C))$
5. $(\neg(A \lor B) \leftrightarrow ((\neg C \land \neg A) \to \neg B))$
6. $((\neg(A \land \neg B) \to C) \land \neg(A \land D))$

For each sentence, find a tautologically equivalent sentence in DNF and one in CNF.